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Philosophical Applications of Modal Logic
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Philosophical Applications of Modal Logic

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Contents

Preface ............................................. ii

1 Modal Logic ........................................ 1
   1.1 Introductory Overview ............................... 1
   1.2 Non-Modal Background ............................... 5
   1.3 Modal Logics: A Hierarchy of Classes ................. 16
   1.4 Refinements and Generalizations ..................... 30

2 Normal Modal Logics ................................. 33
   2.1 Some Candidate Axioms .............................. 33
   2.2 Models and Truth: Simplified Semantics .............. 53
   2.3 Models and Truth: Kripke Semantics .................... 58
   2.4 Canonical Models and Generated Submodels ......... 61
   2.5 From Models to Frames ............................... 71
   2.6 The Rule of Disjunction ............................. 112
   2.7 Revision Exercises ................................ 129
   2.8 Supplement: Matsumoto’s Embedding .................. 142
   2.9 Supplement: Matsumoto’s Embedding (Concluded) .... 149
   2.10 Semantical Postscript ............................... 152
       A Quick History ................................ 152
       Frames with Functions ............................. 157
       An Application of Neighbourhood Semantics ....... 171

3 Applications: Tense Logic ............................. 177
   3.1 Axiomatizing the Basic Logic ....................... 177
   3.2 Extensions of \(K_t\) ................................ 183
   3.3 Temporally Motivated Concerns: Density and Discreteness .. 194

4 Applications: Alethic, Nomic, Deontic .................. 203
   4.1 Introduction ...................................... 203
   4.2 Nomic Necessity I: Pargetter ....................... 217
   4.3 Nomic Necessity II: Bacon ........................... 226
   4.4 Deontic Logic: Main Themes .......................... 234
   4.5 Deontic Logic: More Translations, More Issues .... 278
   4.6 Logics Which Are Fully Modalized ................... 290
   4.7 “Nothing in Between”: A Remark by A. N. Prior .... 304
   4.8 The Fatal Disjunction: Danielsson on Ross’s Paradox .. 324
# CONTENTS

5 Applications: Doxastic and Epistemic Logic .................................................. 331
   5.1 The Logical Omniscience Issue ............................................................... 331
   5.2 Introspection Issues ................................................................................... 370
   5.3 Negative Introspection: Van der Hoek et al. ................................................. 380
   5.4 Negative Introspection: Halpern ................................................................. 385
   5.5 Logics Between $\mathbf{S4}$ and $\mathbf{S5}$ .............................................................. 402
   5.6 Concept-Possession Problems ...................................................................... 420
   5.7 Another Use for $B^{-1}$ ................................................................................. 441

6 Coming to Stand in a Relation  ........................................................................... 449
   6.1 Introduction .................................................................................................. 449
   6.2 The Logic of Coming About ........................................................................ 452
   6.3 Relational Change and Coming About ......................................................... 456
   6.4 Multiple Inchoativity .................................................................................. 459
   6.5 Cross-Predicability ..................................................................................... 460
   6.6 Cross-Predicability and Multiple Inchoativity ............................................. 467
   6.7 Postscript: Completeness for the Logic of §6.2. ........................................... 468

7 Appendix: Natural Deduction for $\mathbf{S4}$ and $\mathbf{S5}$ ......................................... 471
   7.1 Non-Modal Rules ....................................................................................... 471
   7.2 Natural Deduction Rules for $\Box$ ............................................................... 475
   7.3 Adding Rules for $\Diamond$ .......................................................................... 478
   7.4 The Semantics of $\mathbf{S5}$ and Some Rules for $\mathbf{S4}$ ..................................... 481
   7.5 Natural Deduction for Weaker Normal Modal Logics .................................... 486

References ............................................................................................................. 494

Index ...................................................................................................................... 554
Philosophical Applications of Modal Logic
Preface

Modal logic, especially the range of what are called normal modal logics, taken collectively, has been applied to many areas of philosophical interest: to the representation of moral claims and principles (‘deontic logic’), to questions of knowledge and belief (‘epistemic logic’), and so on. It is to such philosophically motivated uses of modal logic – and specifically modal propositional logic (modal predicate being mentioned only in passing from time to time) – that the present work is devoted, though naturally the purely formal elaboration of the subject will require some attention first, before candidate applications can be reviewed. No fixed line is taken on the plausibility of the applications in question, but those wanting to make up their own minds will have many of the pertinent issues aired here in order to help them do so, together with plenty of references to the relevant literature. This should give the general idea of what is covered, though the following paragraph goes into more detail. In particular, while taking a great interest in the application of modal logic to various areas of philosophy, this book pays little attention to the philosophy of modal logic (or philosophy of logic more generally) and none to the philosophy (the metaphysics and epistemology, that is) of modality. The latter area has been treated extensively in many works, some of them (such as Forbes [319] and Lewis [726]) already in our list of references because of the logical issues they touch on, as well as some mentioned only here to redirect readers wanting such material to them: Jubien [633], and – for a more logic-oriented approach – Rini and Cresswell [959], as well as Stalnaker [1084], or Hale and Hoffmann [425] for a useful anthology.

Chapters 1 and 2 give the necessary technical background on the taxonomy of modal logics, their semantic treatment, especially for the normal modal logics, completeness and (what is called ‘global’) modal definability, the rule of disjunction and variations, etc., again, mostly as they arise for normal modal logics. Indeed somewhat more is provided than the bare technical prerequisites for subsequent chapters, when matters of logical interest arise naturally in the course of supplying such a background. And further, some technical matters are found in the later chapters when the application under discussion naturally calls for them or for this or that reason are most conveniently discussed there. For example, local (as opposed to global) modal definability is addressed in the ‘Applications’ chapter on tense logic – more specifically, in Section 3.2 – while a brief introduction to two-dimensional modal semantics can be found in the course of the discussion of deontic logic (after Example 4.4.45), in a chapter which also considers (Section 4.6) the issue of what have been called fully modalized logics, we well as in Section 5.7. Similarly, quite a bit of the discussion of non-normal modal logics is deferred to a point (p. 360ff.) at which it becomes particularly relevant – in the discussion of deductive omniscience in epistemic logic (Section 5.1). Moral: in some cases the index will be of greater assistance than the contents pages for locating a topic of interest. It goes without saying, perhaps, that the topics chosen for discussion here reflect the author’s own interests, which are in broadly philosophical and conceptual matters, and those wanting material on, for example, questions of computational complexity, should look elsewhere. The same goes for those wanting an overview of the last thirty years’ advances in technical modal logic.

Further prefatory material on notation, pre-requisites, etc., can be found at the end of Section 1.1; this includes suggestion for reading in the areas just described as not lying on the present agenda. In addition, I should mention that philosophically motivated departures from the standard language with one or more primitive one-place modal operators, such as the two-place conditional obligation operator often urged in deontic logic, are remarked on in passing but not given a detailed treatment – though plenty of pointers to the pertinent literature are supplied on such occasions.

This material started life as a lecture course in modal logic delivered to philosophy undergraduates at Princeton University in 2007. I am grateful to the Princeton Philosophy Department for inviting me to visit on that occasion and to those students who participated in the course for their feedback and their questions. I have since also had the benefit of assistance on this project from many people, including Rohan French, Allen Hazen, Wolfgang Lenzen, David Makinson and Evgeny Zolin, all of whom provided information or advice on specific topics. For their proof-reading assistance, I am greatly indebted to Rohan French and Sam Butchart.