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Centers of Quantum-Wajsberg Algebras

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Abstract

We define the Wajsberg-center and the OML-center of a quantum-Wajsberg algebra, and study their structures. We prove that the Wajsberg-center is a Wajsberg subalgebra of a quantum-Wajsberg algebra, and that it is a distributive sublattice of its corresponding poset. If the quantum-Wajsberg algebra is quasi-linear, we show that the Wajsberg-center is a linearly ordered Wajsberg algebra. We also show that the lattice subreduct of the Wajsberg-center is a Kleene algebra. Furthermore, we prove that the OML-center is an orthomodular lattice, and that the orthomodular lattices form a subvariety of the variety of quantum-Wajsberg algebras.

1 Introduction

In the last decades, the study of algebraic structures related to the logical foundations of quantum mechanics became a central topic of research. Generally known as quantum structures, these algebras serve as algebraic semantics for the classical and non-classical logics, as well as for the quantum logics. As algebraic structures connected with quantum logics we mention the following algebras: bounded involutive lattices, De Morgan algebras, ortholattices, orthomodular lattices, MV algebras, quantum MV algebras.

The quantum-MV algebras (or QMV algebras) were introduced by R. Giuntini in [7] as non-lattice generalizations of MV algebras ([3]) and as non-idempotent generalizations of orthomodular lattices ([1, 26]). These structures were intensively studied by R. Giuntini ([8, 9, 10, 11, 12]), A. Dvurečenskij and S. Pulmannová ([5]), R. Giuntini and S. Pulmannová ([13]) and by A. Iorgulescu in [20, 21, 22, 23, 24, 25]. An extensive study on the orthomodular structures as quantum logics can be found

The author is very grateful to the anonymous referees for their useful remarks and suggestions on the subject that help improve the presentation.

in [31]. Many algebraic semantics for the classical and non-classical logics studied so far (pseudo-effect algebras, residuated lattices, pseudo-MV/BL/MTL algebras, bounded non-commutative Rℓ-monoids, pseudo-hoops, pseudo-BCK/BCI algebras), as well as their commutative versions, are quantum-B algebras.

Quantum-B algebras, defined and investigated by W. Rump and Y.C. Yang ([33, 32]), arise from the concept of quantales which was introduced in 1984 as a framework for quantum mechanics with a view toward non-commutative logic ([29]). Interesting results on quantum-B algebras have been presented in [34, 35, 15, 16].

We redefined in [4] the quantum-MV algebras starting from involutive BE algebras and we introduced and studied the notion of quantum-Wajsberg algebras (QW algebras, for short). We proved that any Wajsberg algebra is a quantum-Wajsberg algebra, and the commutative quantum-Wajsberg algebras are Wajsberg algebras. It was also shown that the Wajsberg algebras are both quantum-Wajsberg algebras and commutative quantum-B algebras.

In this paper, we define the Wajsberg-center or the commutative center of a quantum-Wajsberg algebra X as the set of those elements of X that commute with all other elements of X. We study certain properties of the Wajsberg-center, and we prove that the Wajsberg-center is a Wajsberg subalgebra of X, and it is also a distributive sublattice of its corresponding poset. If the quantum-Wajsberg algebra is quasi-linear, we show that the Wajsberg-center is a linearly ordered Wajsberg algebra. We also prove that the lattice subreduct of the Wajsberg-center is a Kleene algebra. Furthermore, we define the OML-center of a quantum-Wajsberg algebra, and study its properties. We prove that the OML-center is an orthomodular lattice, and that the orthomodular lattices form a subvariety of the variety of quantum-Wajsberg algebras. Additionally, we prove new properties of quantum-Wajsberg algebras.

2 Preliminaries

In this section, we recall some basic notions and results regarding BCK algebras, Wajsberg algebras, BE algebras and quantum-Wajsberg algebras that will be used in the paper. Additionally, we prove new properties of quantum-Wajsberg algebras. For more details regarding the quantum-Wajsberg algebras we refer the reader to [4].

Starting from the systems of positive implicational calculus, weak systems of positive implicational calculus and BCI and BCK systems, in 1966 Y. Imai and K. Isèki introduced the *BCK algebras* ([17]). BCK algebras are also used in a dual form, with an implication $\rightarrow$ and with one constant element 1, that is the greatest element

([28]). A (dual) BCK algebra is an algebra $(X, \rightarrow, 1)$ of type $(2, 0)$ satisfying the following conditions, for all $x, y, z \in X$: (BCK_1) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$; (BCK_2) $1 \rightarrow x = x$; (BCK_3) $x \rightarrow 1 = 1$; (BCK_4) $x \rightarrow y = 1$ and $y \rightarrow x = 1$ imply $x = y$. In this paper, we use the dual BCK algebras. If $(X, \rightarrow, 1)$ is a BCK algebra, for $x, y \in X$ we define the relation $\leq$ by $x \leq y$ if and only if $x \rightarrow y = 1$, and $\leq$ is a partial order on X.

Wajsberg algebras were introduced in 1984 by Font, Rodriguez and Torrens in [6] as algebraic model of $\aleph_0$-valued Łukasiewicz logic. A *Wajsberg algebra* is an algebra $(X, \rightarrow, ^*, 1)$ of type $(2, 1, 0)$ satisfying the following conditions for all $x, y, z \in X$: (W_1) $1 \rightarrow x = x$; (W_2) $(y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = 1$; (W_3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$; (W_4) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$. Wajsberg algebras are bounded with $0 = 1^*$, and they are involutive. It was proved in [6] that Wajsberg algebras are termwise equivalent to MV algebras.

BE algebras were introduced in [27] as algebras $(X, \rightarrow, 1)$ of type $(2, 0)$ satisfying the following conditions, for all $x, y, z \in X$: (BE_1) $x \rightarrow x = 1$; (BE_2) $x \rightarrow 1 = 1$; (BE_3) $1 \rightarrow x = x$; (BE_4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$. A relation $\leq$ is defined on X by $x \leq y$ iff $x \rightarrow y = 1$. A BE algebra X is *bounded* if there exists $0 \in X$ such that $0 \leq x$, for all $x \in X$. In a bounded BE algebra $(X, \rightarrow, 0, 1)$ we define $x^* = x \rightarrow 0$, for all $x \in X$. A bounded BE algebra X is called *involutive* if $x^{**} = x$, for any $x \in X$.
A BE algebra X is called *commutative* if $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for all $x, y \in X$. A bounded BE algebra X is called *involutive* if $x^{**} = x$, for any $x \in X$.
Obviously, any BCK algebra is a BE algebra, but the exact connection between BE algebras and BCK algebras is made in the papers [18, 19]: a BCK algebra is a BE algebra satisfying (BCK_4) (antisymmetry) and (BCK_1).

A *suplement algebra* (*S-algebra, for short*) is an algebra $(X, \oplus, ^*, 0, 1)$ of type $(2, 1, 0, 0)$ satisfying the following axioms for all $x, y, z \in X$: (S_1) $x \oplus y = y \oplus x$; (S_2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$; (S_3) $x \oplus x^* = 1$; (S_4) $x \oplus 0 = x$; (S_5) $x^{**} = x$; (S_6) $0^* = 1$; (S_7) $x \oplus 1 = 1$ ([14]).
The following additional operations can be defined in a supplement algebra:
$x \odot y = (x^* \oplus y^*)^*$, $x \Cap_S y = (x \oplus y^*) \odot y$, $x \Cup_S y = (x \odot y^*) \oplus y$.
A *quantum-MV algebra* (*QMV algebra, for short*) is an S-algebra $(X, \oplus, ^*, 0, 1)$ satisfying the following axiom for all $x, y, z \in X$ ([8]):
(QMV) $x \oplus ((x^* \Cap_S y) \Cap_S (z \Cap_S x^*)) = (x \oplus y) \Cap_S (x \oplus z)$.

Lemma 2.1. *Let $(X, \rightarrow, 1)$ be a BE algebra. The following hold for all $x, y, z \in X$:*
(1) $x \rightarrow (y \rightarrow x) = 1$;
(2) $x \leq (x \rightarrow y) \rightarrow y$.
If X is bounded, then:

(3) $x \to y^* = y \to x^*$;
(4) $x \leq x^{**}$.
If X is involutive, then:
(5) $x^* \to y = y^* \to x$;
(6) $x^* \to y^* = y \to x$;
(7) $(x \to y)^* \to z = x \to (y^* \to z)$;
(8) $x \to (y \to z) = (x \to y^*)^* \to z$;
(9) $(x^* \to y)^* \to (x^* \to y) = (x^* \to x)^* \to (y^* \to y)$.

Proof. (1)-(6) See [4].
(7) Applying (BE_4) we get: $(x \to y)^* \to z = z^* \to (x \to y) = x \to (z^* \to y) = x \to (y^* \to z)$.
(8) Using (BE_4), we have: $x \to (y \to z) = x \to (z^* \to y^*) = z^* \to (x \to y^*) = (x \to y^*)^* \to z$.
(9) Applying twice (7), we get: $(x^* \to y)^* \to (x^* \to y) = x^* \to (y^* \to (x^* \to y)) = x^* \to (x^* \to (y^* \to y)) = (x^* \to x)^* \to (y^* \to y)$. □

In a BE algebra X, we define the additional operation $x \Cup y = (x \to y) \to y$. If X is involutive, we define the operations $x \Cap y = ((x^* \to y^*) \to y^*)^* = (x^* \Cup y^*)^*$, $x \odot y = (x \to y^*)^* = (y \to x^*)^*$, and the relation $\leq_Q$ by $x \leq_Q y$ iff $x = x \Cap y$.

Proposition 2.2. *Let X be an involutive BE algebra. Then the following hold for all $x, y, z \in X$:*
(1) $x \leq_Q y$ *implies* $x \leq y$, $x = y \Cap x$ *and* $y = x \Cup y$;
(2) $\leq_Q$ *is reflexive and antisymmetric;*
(3) $(x \Cap y) \to z = (y \to x) \to (y \to z)$;
(4) $(x \Cap y)^* \to (y \to x)^* = y \Cup (y \to x)^*$;
(5) $(x \Cap (y \Cap z))^* = ((z \to x) \Cap (z \to y)) \to z^*$;
(6) $x, y \leq_Q z$ *and* $z \to x = z \to y$ *imply* $x = y$; (cancellation law)
(7) $x \Cap y = y \odot (y \to x)$.

Proof. (1) – (3) See [4].
(4) We have:

$$\begin{aligned}(x \Cap y)^* \to (y \to x)^* &= ((x^* \to y^*) \to y^*) \to (y \to x)^* \\ &= ((y \to x) \to y^*) \to (y \to x)^* \\ &= (y \to (y \to x)^*) \to (y \to x)^* = y \Cup (y \to x)^*.\end{aligned}$$

(5) Aplying (3), we get:

$$\begin{aligned}((z \to x) \Cap (z \to y)) \to z^* &= ((x^* \to z^*) \Cap (y^* \to z^*)) \to z^* \\ &= ((y^* \to z^*) \to (x^* \to z^*)) \to ((y^* \to z^*) \to z^*) \\ &= (x^* \to ((y^* \to z^*) \to z^*)) \to ((y^* \to z^*) \to z^*)\end{aligned}$$

$= (x^* \to (y^* \Cup z^*)) \to (y^* \Cup z^*)$
$= (x^* \to (y \Cap x)^*) \to (y \Cap z)^*$
$= (x \Cap (y \Cap z))^*$.

(6) Since $x, y \leq_Q z$ and $z \to x = z \to y$, we have:

$$x = x \Cap z = ((x^* \to z^*) \to z^*)^* = ((z \to x) \to z^*)^*$$
$$= ((z \to y) \to z^*)^* = ((y^* \to z^*) \to z^*)^* = y \Cap z = y.$$

(7) We have $y \odot (y \to x) = (y \to (y \to x)^*)^* = ((y \to x) \to y^*)^* = ((x^* \to y^*) \to y^*)^* = x \Cap y$. □

A *(left-)quantum-Wajsberg algebra* (*QW algebra, for short*) $(X, \to, ^*, 1)$ is an involutive BE algebra $(X, \to, ^*, 1)$ satisfying the following condition for all $x, y, z \in X$:

(QW) $x \to ((x \Cap y) \Cap (z \Cap x)) = (x \to y) \Cap (x \to z)$.

Condition (QW) is equivalent to the following conditions:

(QW_1) $x \to (x \Cap y) = x \to y$;

(QW_2) $x \to (y \Cap (z \Cap x)) = (x \to y) \Cap (x \to z)$.

Definition 2.3. (*[20]*) A *(left-)m-BE algebra* is an algebra $(X, \odot, ^*, 1)$ of type $(2, 1, 0)$ satisfying the following properties, for all $x, y, z \in X$: (PU) $1 \odot x = x = x \odot 1$; (Pcomm) $x \odot y = y \odot x$; (Pass) $x \odot (y \odot z) = (x \odot y) \odot z$; (m-L) $x \odot 0 = 0$; (m-Re) $x \odot x^* = 0$, where $0 := 1^*$.

Note that, according to [25, Cor. 17.1.3], the involutive (left-)BE algebras $(X, \to, ^*, 1)$ are term-equivalent to involutive (left-)m-BE algebras $(X, \odot, ^*, 1)$, by the mutually inverse transformations ([20, 25]):

$$\Phi: \ x \odot y := (x \to y^*)^* \quad \text{and} \quad \Psi: \ x \to y := (x \odot y^*)^*.$$

Definition 2.4. (*[24, Def. 3.10]*) A *(left-)quantum-MV algebra*, or a *(left-)QMV algebra* for short, is an involutive (left-)m-BE algebra $(X, \odot, ^*, 1)$ verifying the following axiom: for all $x, y, z \in X$,

(Pqmv) $x \odot ((x^* \Cup y) \Cup (z \Cup x^*)) = (x \odot y) \Cup (x \odot z)$.

Proposition 2.5. *The (left-)quantum-Wajsberg algebras are term-equivalent to (left-)quantum-MV algebras.*

Proof. We prove that the axioms (*Pqmv*) and (*QW*) are equivalent. Using the transformation Φ, from (Pqmv) we get:

$x \odot ((x^* \Cup y) \Cup (z \Cup x^*)) = (x \to ((x^* \Cup y) \Cup (z \Cup x^*))^*)^* = (x \to ((x \Cap y^*) \Cap (z^* \Cap x)))^*$

and

$(x \odot y) \Cup (x \odot z) = (x \to y^*)^* \Cup (x \to z^*)^* = ((x \to y^*) \Cap (x \to z^*))^*$,

hence (Pqmv) becomes:

$(x \to ((x \Cap y^*) \Cap (z^* \Cap x)))^* = ((x \to y^*) \Cap (x \to z^*))^*$,
for all $x, y, z \in X$. Replacing y by y^* and z by z^*, we get axiom (QW). Similarly axiom (QW) implies axiom (Pqmv). □

In what follows, by quantum-MV algebras and quantum-Wajsberg algebras we understand the left-quantum-MV algebras and left-quantum-Wajsberg algebras, respectively.

Proposition 2.6. *([4]) Let X be a quantum-Wajsberg algebra. The following hold for all $x, y, z \in X$:*
(1) $x \to (y \Cap x) = x \to y$ and $(x \to y) \to (y \Cap x) = x$;
(2) $x \leq_Q x^* \to y$ and $x \leq_Q y \to x$;
(3) $x \leq y$ iff $y \Cap x = x$;
(4) $(x \to y) \Cup (y \to x) = 1$.
If $x \leq_Q y$, then:
(5) $y = y \Cup x$;
(6) $y^* \leq_Q x^*$;
(7) $y \to z \leq_Q x \to z$ and $z \to x \leq_Q z \to y$;
(8) $x \Cap z \leq_Q y \Cap z$ and $x \Cup z \leq_Q y \Cup z$;
(9) $x \odot z \leq_Q y \odot z$.

Proposition 2.7. *Let X be a quantum-Wajsberg algebra. The following hold, for all $x, y, z \in X$:*
(1) $(x \Cap y) \Cap (y \Cap z) = (x \Cap y) \Cap z$;
(2) $\leq_Q$ is transitive;
(3) $(z \Cap x) \to (y \Cap x) = (z \Cap x) \to y$;
(4) $x \leq_Q y$ and $y \leq x$ imply $x = y$;
(5) $x \leq_Q y$ implies $x \Cap (y \Cap z) = x \Cap z$;
(6) $z \Cap ((y^* \to z) \Cap (x^* \to y)) = z \Cap (x^* \to y)$;
(7) $x \Cup (x \to y)^* = x$;
(8) $x = y \to x$ iff $y = x \to y$;
(9) $x \Cap y, y \Cap x \leq_Q x \to y$.

Proof. $(1) - (3)$ See [4].
(4) By Proposition 2.6(3), $y \leq x$ implies $x \Cap y = y$. Since $x \leq_Q y$, we have $x \Cap y = x$, hence $x = y$.
(5) Using (1), $(x \Cap y) \Cap (y \Cap z) = (x \Cap y) \Cap z$. Since $x \leq_Q y$ implies $x \Cap y = x$, we get $x \Cap (y \Cap z) = x \Cap z$.
(6) It follows by (5), since $z \leq_Q y^* \to z$;
(7) By Proposition 2.6(2),(5), we have $x^* \leq_Q x \to y$, so that $(x \to y)^* \leq_Q x$ and

$x \Cup (x \to y)^* = x$.
(8) Suppose $x = y \to x$, so that $y^* \Cap x^* = (y \Cup x)^* = ((y \to x) \to x)^* = (x \to x)^* = 1^* = 0$. Using (QW_1), we get $y = (y^*)^* = y^* \to 0 = y^* \to (y^* \Cap x^*) = y^* \to x^* = x \to y$. The converse follows similarly.
(9) Using Proposition 2.6(2), we have $y \leq_Q x \to y$, so that $(x \to y)^* \leq_Q y^* \leq_Q (x^* \to y^*) \to y^* = (x \Cap y)^*$. Hence $x \Cap y \leq_Q x \to y$. Similarly $(x \to y)^* \leq (x \to y) \to x^* = (y^* \to x^*) \to x^* = (y \Cap x)^*$. Thus $y \Cap x \leq_Q x \to y$. □

By Propositions 2.2(2), 2.7(2), in a quantum-Wajsberg algebra X, $\leq_Q$ is a partial order on X.
A quantum-Wajsberg algebra X is called *commutative* if $x \Cup y = y \Cup x$, or equivalently $x \Cap y = y \Cap x$ for all $x, y \in X$.
Since:
- commutative BE algebras are commutative BCK algebras ([36]]),
- bounded commutative BCK are term-equivalent to MV algebras ([30]) and
- Wajsberg algebras are term-equivalent to MV algebras ([6]),
it follows that bounded commutative BE algebras are bounded commutative BCK algebras, hence are term-equivalent to MV algebras, hence to Wajsberg algebras.
Hence the commutative quantum-Wajsberg algebras are the Wajsberg algebras.
It was proved in [4] that a quantum-Wajsberg algebra is a bounded commutative BCK algebra, that is a Wajsberg algebra, if and only if the relations $\leq$ and $\leq_Q$ coincide.

Proposition 2.8. *([4]) Let $(X, \to, 0, 1)$ be a bounded commutative BCK algebra. The following hold for all $x, y, z \in X$:*
(1) $x \leq_Q y$ and $x \leq_Q z$ imply $x \leq_Q y \Cap z$;
(2) $y \leq_Q x$ and $z \leq_Q x$ imply $y \Cup z \leq_Q x$;
(3) $x \leq_Q y$ implies $x \Cup z \leq_Q y \Cup z$ and $x \Cap z \leq_Q y \Cap z$.

3 The Wajsberg-center of quantum-Wajsberg algebras

In this section, we investigate the commutativity property of quantum-Wajsberg algebras. We define the Wajsberg-center or the commutative center of a quantum-Wajsberg algebra X as the set of those elements of X that commute with all other elements of X. We study certain properties of the Wajsberg-center, and prove that the Wajsberg-center is a Wajsberg subalgebra of X. In what follows, $(X, \to, ^*, 1)$ will be a quantum-Wajsberg algebra, unless otherwise stated.

Definition 3.1. We say that the elements $x, y \in X$ *commute*, denoted by $x\mathcal{C}y$, if $x \Cap y = y \Cap x$.

Definition 3.2. The *commutative center* of X is the set $\mathcal{Z}(X) = \{x \in X \mid x\mathcal{C}y$, for all $y \in X\}$.

Obviously $0, 1 \in \mathcal{Z}(X)$.

Lemma 3.3. *If $x\mathcal{C}y$, then $x \Cup y = y \Cup x$.*

Proof. Applying twice Proposition 2.6(1), we have:

$$\begin{aligned} x \Cup y &= (x \to y) \to y = (x \to y) \to ((y \to x) \to (x \Cap y)) \\ &= (y \to x) \to ((x \to y) \to (x \Cap y)) \\ &= (y \to x) \to ((x \to y) \to (y \Cap x)) = (y \to x) \to x = y \Cup x. \end{aligned}$$ □

Lemma 3.4. *Let $x, y \in X$. The following are equivalent:*
(a) $x\mathcal{C}y$;
(b) $(x \to y) \to (x \Cap y) = x$.

Proof. (a) ⇒ (b) By Proposition 2.6(1), we get $x = (x \to y) \to (y \Cap x) = (x \to y) \to (x \Cap y)$.
(b) ⇒ (a) Suppose $(x \to y) \to (x \Cap y) = x$, and applying Proposition 2.6(1), we have: $(x \to y) \to (x \Cap y) = (x \to y) \to (y \Cap x) (= x)$. Since by Proposition 2.7(9), $x \Cap y, y \Cap x \le_Q x \to y$, by cancellation law (Proposition 2.2(6)), we get $x \Cap y = y \Cap x$. Hence $x\mathcal{C}y$. □

Proposition 3.5. *The following hold:*
(1) the relation $\mathcal{C}$ is reflexive and symmetric;
(2) if $x \le_Q y$ or $y \le_Q x$, then $x\mathcal{C}y$;
(3) $x\mathcal{C}y$ implies $x^*\mathcal{C}y^*$;
(4) $(x \Cap y)^*\mathcal{C}(x \to y)^*$.

Proof. (2) If $x \le_Q y$, then $x = x \Cap y$ and, by Proposition 2.2(1) we have $x = y \Cap x$. Hence $x \Cap y = y \Cap x$, that is $x\mathcal{C}y$, and similarly $y \le_Q x$ implies $x\mathcal{C}y$.
(3) Using Lemma 3.3, we have: $x^* \Cap y^* = (x \Cup y)^* = (y \Cup x)^* = y^* \Cap x^*$, hence $x^*\mathcal{C}y^*$.
(4) Since $x \Cap y \le_Q y \le_Q x \to y$, we get $(x \to y)^* \le_Q (x \Cap y)^*$. Applying (2), it follows that $(x \Cap y)^*\mathcal{C}(x \to y)^*$. □

Corollary 3.6. *$\mathcal{Z}(X)$ is closed under *.*

Proof. Let $x \in \mathcal{Z}(X)$, that is $x\mathcal{C}z$ for all $z \in X$. We also have $x\mathcal{C}z^*$, and applying Lemma 3.3 we have $x \Cup z^* = z^* \Cup x$. It follows that $x^* \Cap z = (x \Cup z^*)^* = (z^* \Cup x)^* = z \Cap x^*$. Hence $x^* \in \mathcal{Z}(X)$. □

Proposition 3.7. *If $x, y, z \in X$ such that $x\mathcal{C}y$ and $x\mathcal{C}z$, then $(x \Cap y) \Cap z = y \Cap (x \Cap z)$.*

Proof. From $x \Cap y = y \Cap x$, $x \Cap z = z \Cap x$, and applying Proposition 2.7(1),(3), we get:

$$\begin{aligned}(x \Cap y) \Cap z &= (y \Cap x) \Cap z = (y \Cap x) \Cap (x \Cap z)\\ &= (((y \Cap x)^* \to (x \Cap z)^*) \to (x \Cap z)^*)^*\\ &= (((y \Cap x)^* \to (z \Cap x)^*) \to (z \Cap x)^*)^*\\ &= (((z \Cap x) \to (y \Cap x)) \to (z \Cap x)^*)^*\\ &= (((z \Cap x) \to y) \to (z \Cap x)^*)^*\\ &= ((y^* \to (z \Cap x)^*) \to (z \Cap x)^*)^*\\ &= y \Cap (z \Cap x) = y \Cap (x \Cap z).\end{aligned}$$

□

Corollary 3.8. *If $x\mathcal{C}y$, $y\mathcal{C}z$ and $x\mathcal{C}z$, then $(x \Cap y) \Cap z = z \Cap (x \Cap y)$.*

Proof. By hypothesis and using Proposition 3.7, we get:

$$(x \Cap y) \Cap z = y \Cap (x \Cap z) = y \Cap (z \Cap x) = z \Cap (y \Cap x) = z \Cap (x \Cap y).$$

□

Corollary 3.9. *$\mathcal{Z}(X)$ is closed under $\Cap$.*

Proof. Let $x, y \in \mathcal{Z}(X)$ and let $z \in X$. It follows that $x\mathcal{C}y$, $y\mathcal{C}z$, $x\mathcal{C}z$, and by Corollary 3.8 we get $(x \Cap y) \Cap z = z \Cap (x \Cap y)$. Hence $x \Cap y \in \mathcal{Z}(X)$, that is $\mathcal{Z}(X)$ is closed under $\Cap$. □

Proposition 3.10. *Let $x, y, z \in X$ such that $y\mathcal{C}z$. Then $x \to (y \Cap z) \leq_Q (x \to y) \Cap (x \to z)$.*

Proof. From $z \Cap y \leq_Q y$, we get $x \to (z \Cap y) \leq_Q x \to y$, so that $(x \to y)^* \leq_Q (x \to (z \Cap y))^*$ and $(x \to (z \Cap y))^* \to (x \to z)^* \leq_Q (x \to y)^* \to (x \to z)^*$. It follows that:

$$((x \to (z \Cap y))^* \to (x \to z)^*) \Cap ((x \to y)^* \to (x \to z)^*) = (x \to (z \Cap y))^* \to (x \to z)^*.$$

Similarly, from $y \Cap z \leq_Q z$ we have $x \to (y \Cap z) \leq_Q x \to z$, hence $(x \to (y \Cap z)) \Cap (x \to z) = x \to (y \Cap z)$. Applying Proposition 2.2(5), and taking into consideration that $y\mathcal{C}z$, we have:

$$\begin{aligned}&(x \to (y \Cap z)) \Cap ((x \to y) \Cap (x \to z)) =\\ &\quad = ((((x \to z) \to (x \to (y \Cap z))) \Cap ((x \to z) \to (x \to y))) \to (x \to z)^*)^*\\ &\quad = ((((x \to (y \Cap z))^* \to (x \to z)^*) \Cap ((x \to y)^* \to (x \to z)^*)) \to (x \to z)^*)^*\\ &\quad = ((((x \to (z \Cap y))^* \to (x \to z)^*) \Cap ((x \to y)^* \to (x \to z)^*)) \to (x \to z)^*)^*\\ &\quad = (((x \to (z \Cap y))^* \to (x \to z)^*) \to (x \to z)^*)^*\\ &\quad = (((x \to (y \Cap z))^* \to (x \to z)^*) \to (x \to z)^*)^*\\ &\quad = (x \to (y \Cap z)) \Cap (x \to z) = x \to (y \Cap z).\end{aligned}$$

Hence $x \to (y \Cap z) \leq_Q (x \to y) \Cap (x \to z)$. □

Lemma 3.11. *If $x\mathcal{C}y$, $x\mathcal{C}z$, $y\mathcal{C}z$, then $y \Cap (z \Cap x) \le_Q y \Cap z$.*

Proof. From $z \Cap x = x \Cap z \le_Q z$, by Proposition 2.6(8) we get $(z \Cap x) \Cap y \le_Q z \Cap y = y \Cap z$. Using Corollary 3.8, we get $y \Cap (z \Cap x) \le_Q y \Cap z$. ☐

Proposition 3.12. *If $x\mathcal{C}y$, $x\mathcal{C}z$, $y\mathcal{C}z$, then $(x \to y) \Cap (x \to z) \le x \to (y \Cap z)$.*

Proof. Applying Proposition 2.2(3) and Lemma 3.11, we get:

$$\begin{aligned} ((x \to y) \Cap (x \to z)) \to (x \to (y \Cap z)) = \\ &= (x \to (y \Cap z))^* \to ((x \to y) \Cap (x \to z))^* \\ &= (x \to (y \Cap z))^* \to ((x \to y)^* \Cup (x \to z)^*) \\ &= (x \to (y \Cap z))^* \to (((x \to y)^* \to (x \to z)^*) \to (x \to z)^*) \\ &= ((x \to y)^* \to (x \to z)^*) \to ((x \to (y \Cap z))^* \to (x \to z)^*) \\ &= ((x \to z) \to (x \to y)) \to ((x \to z) \to (x \to (y \Cap z))) \\ &= ((z \Cap x) \to y) \to ((z \Cap x) \to (y \Cap z)) \\ &= (y \Cap (z \Cap x)) \to (y \Cap z) = 1. \end{aligned}$$

It follows that $(x \to y) \Cap (x \to z) \le x \to (y \Cap z)$. ☐

Proposition 3.13. *If $x\mathcal{C}y$, $x\mathcal{C}z$, $y\mathcal{C}z$, then $x \to (y \Cap z) = (x \to y) \Cap (x \to z)$.*

Proof. It follows by Propositions 3.10, 3.12, 2.7(4). ☐

Corollary 3.14. *If $y \in \mathcal{Z}(X)$ and $x, z \in X$, then $(z \Cap x)^* \to ((z \to x)^* \Cap y) = ((z \Cap x)^* \to (z \to x)^*) \Cap ((z \Cap x)^* \to y)$.*

Proof. It follows by Propositions 3.5(4) and 3.13, since $y\mathcal{C}(z \Cap x)^*$ and $y\mathcal{C}(z \to x)^*$. ☐

Corollary 3.15. *If $x, y, z \in \mathcal{Z}(X)$, then $(z \Cap x)^* \to y = (y^* \to z) \Cap (x^* \to y)$.*

Proof. Since $y \in \mathcal{Z}(X)$ implies $y^* \in \mathcal{Z}(X)$, applying Proposition 3.13, we get: $(z \Cap x)^* \to y = y^* \to (z \Cap x) = (y^* \to z) \Cap (y^* \to x) = (y^* \to z) \Cap (x^* \to y)$. ☐

Proposition 3.16. *If $x, y \in \mathcal{Z}(X)$, then $x^* \to y \in \mathcal{Z}(X)$.*

Proof. Let $x, y \in \mathcal{Z}(X)$ and let $z \in X$. Then $x\mathcal{C}z$ and $y\mathcal{C}(z \to x)^*$. Applying Lemma 3.4, we get:

$$z = (z \to x) \to (z \Cap x) = ((z \to x)^*)^* \to (z \Cap x) \text{ and}$$
$$(z \to x)^* = ((z \to x)^* \to y) \to ((z \to x)^* \Cap y),$$

respectively. It follows that:

$$\begin{aligned} z &= ((z \to x)^*)^* \to (z \Cap x) \\ &= (((z \to x)^* \to y) \to ((z \to x)^* \Cap y))^* \to (z \Cap x) \\ &= ((z \to x)^* \to y) \to (((z \to x)^* \Cap y)^* \to (z \Cap x)) \text{ (by Lemma 2.1(7))} \end{aligned}$$

$= ((z \to x)^* \to y) \to ((z \Cap x)^* \to ((z \to x)^* \Cap y))$
$= ((z \to x)^* \to y) \to (((z \Cap x)^* \to (z \to x)^*) \Cap ((z \Cap x)^* \to y))$ (by Corollary 3.14)
$= ((z \to x)^* \to y) \to (((x \Cap z)^* \to (z \to x)^*) \Cap ((z \Cap x)^* \to y))$
$= ((z \to x)^* \to y) \to ((z \Cup (z \to x)^*) \Cap ((z \Cap x)^* \to y))$ (by Proposition 2.2(4))
$= ((z \to x)^* \to y) \to (z \Cap ((z \Cap x)^* \to y))$ (by Proposition 2.7(7))
$= ((z \to x)^* \to y) \to (z \Cap ((y^* \to z) \Cap (x^* \to y))$ (by Corrolary 3.15)
$= ((z \to x)^* \to y) \to (z \Cap (x^* \to y))$ (by Proposition 2.7(6))
$= (z \to (x^* \to y)) \to (z \Cap (x^* \to y))$ (by Lemma 2.1(7)).
Using Lemma 3.4, we conclude that $x^* \to y \in \mathcal{Z}(X)$. □

Corollary 3.17. *If $x, y \in \mathcal{Z}(X)$, then $x \to y \in \mathcal{Z}(X)$.*

Proof. Since $x \in \mathcal{Z}(X)$, by Corollary 3.6 we get $x^* \in \mathcal{Z}(X)$. Applying Proposition 3.16, $x^*, y \in \mathcal{Z}(X)$ implies $x \to y \in \mathcal{Z}(X)$. □

Theorem 3.18. *$(\mathcal{Z}(X), \to, 0, 1)$ is a Wajsberg subalgebra of X.*

Proof. Since by Corollary 3.17, $x, y \in \mathcal{Z}(X)$ implies $x \to y \in \mathcal{Z}(X)$, it follows that $\mathcal{Z}(X)$ is closed under $\to$. Moreover $0, 1 \in \mathcal{Z}(X)$, hence it is a quantum-Wajsberg subalgebra of X. Since $x, y \in \mathcal{Z}(X)$ implies $x \Cap y = y \Cap x$, $(\mathcal{Z}(X), \to, 0, 1)$ is a commutative quantum-Wajsberg algebra, that is a bounded commutative BCK subalgebra of X. Hence it is a Wajsberg subalgebra of X. □

Corollary 3.19. *A quantum-Wajsberg algebra X is a Wajsberg algebra if and only if $\mathcal{Z}(X) = X$.*

Taking into consideration the above results, the commutative center $\mathcal{Z}(X)$ will be also called the *Wajsberg-center* of X. Similarly as in [7] for the case of QMV algebras, we define the notion of a quasi-linear quantum-Wajsberg algebra.

Definition 3.20. A QW algebra X is said to be *quasi-linear* if, for all $x, y \in X$, $x \nleq_Q y$ implies $y < x$.

Proposition 3.21. *If X is a quasi-linear QW algebra, then $\mathcal{Z}(X)$ is a linearly ordered Wajsberg algebra.*

Proof. According to [4], a quantum-Wajsberg algebra is a Wajsberg algebra if and only if the relations $\le$ and $\le_Q$ coincide. Since $\mathcal{Z}(X)$ is a quasi-linear Wajsberg algebra, $x \nleq y$ implies $y < x$, that is $\mathcal{Z}(X)$ is linearly ordered. □

4 The lattice structure of Wajsberg-centers

We study certain lattice properties of the Wajsberg-center of a quantum-Wajsberg algebra, and prove that the Wajsberg-center of a quantum-Wajsberg algebra X is a distributive sublattice of the poset $(X, \leq_Q, 0, 1)$. If the quantum-Wajsberg algebra is quasi-linear, we prove that the Wajsberg-center is a linearly ordered Wajsberg algebra. Finally, we show that the lattice subreduct of the Wajsberg-center is a Kleene algebra. In what follows, $(X, \rightarrow, ^*, 1)$ will be a quantum-Wajsberg algebra, unless otherwise stated.

Proposition 4.1. *The following hold for all $x, y, z \in \mathcal{Z}(X)$:*
(1) $x \rightarrow (y \Cap z) = (x \rightarrow y) \Cap (x \rightarrow z)$ (*distributivity of* $\rightarrow$ *over* $\Cap$);
(2) $x \odot (y \Cup z) = (x \odot y) \Cup (x \odot z)$ (*distributivity of* $\odot$ *over* $\Cup$);
(3) $x \Cap (y \Cup z) = (x \Cap y) \Cup (x \Cap z)$ (*distributivity of* $\Cap$ *over* $\Cup$);
(4) $x \Cup (y \Cap z) = (x \Cup y) \Cap (x \Cup z)$ (*distributivity of* $\Cup$ *over* $\Cap$).

Proof. (1) It follows by Proposition 3.13.
(2) Applying (1), we get:

$$\begin{aligned} x \odot (y \Cup z) &= (x \rightarrow (y \Cup z)^*)^* = (x \rightarrow (y^* \Cap z^*))^* \\ &= ((x \rightarrow y^*) \Cap (x \rightarrow z^*))^* = (x \rightarrow y^*)^* \Cup (x \rightarrow z^*)^* \\ &= (x \odot y) \Cup (x \odot z). \end{aligned}$$

(3) By commutativity we have $y, z \leq_Q y \Cup z$, so that $(y \Cup z) \rightarrow x \leq_Q y \rightarrow x, z \rightarrow x$. Applying Propositions 2.6(9) and 2.2(7), we get:

$$y \odot ((y \Cup z) \rightarrow x) \leq_Q y \odot (y \rightarrow x) = x \Cap y \text{ and}$$
$$z \odot ((y \Cup z) \rightarrow x) \leq_Q z \odot (z \rightarrow x) = x \Cap z.$$

Using Proposition 2.2(7) and (2), we have:

$$\begin{aligned} x \Cap (y \Cup z) &= (y \Cup z) \odot ((y \Cup z) \rightarrow x) = (y \odot ((y \Cup z) \rightarrow x)) \Cup (z \odot ((y \Cup z) \rightarrow x)) \\ &\leq_Q (x \Cap y) \Cup (x \Cap z). \end{aligned}$$

On the other hand, $x \Cap y \leq_Q y$, $x \Cap z \leq_Q z$ imply $(x \Cap y) \Cup (x \Cap z) \leq_Q y \Cup z$, and $x \Cap y \leq_Q x$, $x \Cap z \leq_Q x$ imply $(x \Cap y) \Cup (x \Cap z) \leq_Q x$. Hence by Proposition 2.8, $(x \Cap y) \Cup (x \Cap z) \leq_Q x \Cap (y \Cup z)$. Since $\mathcal{Z}(X)$ is a commutative bounded BCK algebra, the relation $\leq_Q$ is antisymmetric, and we conclude that $x \Cap (y \Cup z) = (x \Cap y) \Cup (x \Cap z)$.
(4) Applying (3), we have:

$$\begin{aligned} x \Cup (y \Cap z) &= (x^* \Cap (y \Cap z)^*)^* = (x^* \Cap (y^* \Cup z^*))^* \\ &= ((x^* \Cap y^*) \Cup (x^* \Cap z^*))^* = ((x \Cup y)^* \Cup (x \Cup z)^*)^* \\ &= (x \Cup y) \Cap (x \Cup z). \end{aligned}$$

□

Lemma 4.2. *The following hold for all $x, y \in \mathcal{Z}(X)$:*
(1) *$x \Cup y$ is the least upper bound (l.u.b.) of $\{x, y\}$;*
(2) *$x \Cap y$ is the greatest lower bound (g.l.b.) of $\{x, y\}$.*

Proof. (1) By Corollaries 3.9 and 3.6, $\mathcal{Z}(X)$ is closed under $\Cap$ and *. Since $x \Cup y = (x^* \Cap y^*)^*$ for all $x, y \in \mathcal{Z}(X)$, it follows that $\mathcal{Z}(X)$ is also closed under $\Cup$. Since by commutativity $x, y \le_Q x \Cup y$, it follows that $x \Cup y$ is an upper bound of $\{x, y\}$. Let z be another upper bound of $\{x, y\}$, so that $x, y \le_Q z$, that is $x = x \Cap z$ and $y = y \Cap z$. Using Proposition 4.1(3), we have $(x \Cup y) \Cap z = z \Cap (x \Cup y) = (z \Cap x) \Cup (z \Cap y) = (x \Cap z) \Cup (y \Cap z) = x \Cup y$. Hence $x \Cup y \le_Q z$, so that $x \Cup y$ is the l.u.b. of $\{x, y\}$.
(2) By commutativity we also have $x \Cap y \le_Q x, y$, thus $x \Cap y$ is a lower bound of $\{x, y\}$. Let z be another lower bound of $\{x, y\}$, so that $z \le_Q x, y$, that is $z = z \Cap x$ and $z = z \Cap y$. Using Proposition 3.7 and Corollary 3.8, we have: $z \Cap (x \Cap y) = (x \Cap y) \Cap z = y \Cap (x \Cap z) = y \Cap (z \Cap x) = y \Cap z = z \Cap y = z$, that is $z \le_Q x \Cap y$. It follows that $x \Cap y$ is the g.l.b. of $\{x, y\}$. $\square$

Theorem 4.3. *$(\mathcal{Z}(X), \Cap, \Cup, 0, 1)$ is a distributive sublattice of the poset $(X, \le_Q, 0, 1)$.*

Proof. It follows by Lemma 4.2, Theorem 3.18 and Proposition 4.1. $\square$

Proposition 4.4. *The following hold for all $x, y, z \in \mathcal{Z}(X)$:*
(1) $x \to (y \Cup z) = (x \to y) \Cup (x \to z)$ (*distributivity of $\to$ over $\Cup$*);
(2) $x \odot (y \Cap z) = (x \odot y) \Cap (x \Cup z)$ (*distributivity of $\odot$ over $\Cap$*);
(3) $(y \Cup z) \to x = (y \to x) \Cap (z \to x)$;
(4) $(y \Cap z) \to x = (y \to x) \Cup (z \to x)$.

Proof. (1) Since by commutativity $y \Cup z \ge_Q y, z$, we have $x \to (y \Cup z) \ge_Q x \to y, x \to z$, so that $x \to (y \Cup z)$ is an upper bound of $\{x \to y, x \to z\}$. Let u be another upper bound of $\{x \to y, x \to z\}$, that is $u \ge_Q x \to y, x \to z$. It follows that $u \odot x \ge_Q (x \to y) \odot x = x \Cap y$ and $u \odot x \ge_Q (x \to z) \odot x = x \Cap z$. Hence, by Proposition 4.1(3), $x \Cap (y \Cup z) = (x \Cap y) \Cup (x \Cap z) \le_Q u \odot x$. Using (QW_1), we get $x \to (y \Cup z) = x \to (x \Cap (y \Cup z)) \le_Q x \to (u \odot x) = x \to (u \to x^*)^* = (u \to x^*) \to x^* = u \Cup x^* = u$ (since $u \ge_Q x \to y \ge_Q x^*$). Thus $x \to (y \Cup z)$ is the least upper bound of $\{x \to y, x \to z\}$, and so $x \to (y \Cup z) = (x \to y) \Cup (x \to z)$.
(2) Using (1), we have:
$$\begin{aligned} x \odot (y \Cap z) &= (x \to (y \Cap z)^*)^* = (x \to (y^* \Cup z^*))^* \\ &= ((x \to y^*) \Cup (x \to z^*))^* = ((x \odot y)^* \Cup (x \odot z)^*)^* \\ &= (x \odot y) \Cap (x \odot z). \end{aligned}$$
(3) Applying Proposition 4.1(1), we have:
$$\begin{aligned} (y \Cup z) \to x &= x^* \to (y \Cup z)^* = x^* \to (y^* \Cap z^*) \\ &= (x^* \to y^*) \Cap (x^* \to z^*) = (y \to x) \Cap (z \to x). \end{aligned}$$
(4) By (1), we get:
$$\begin{aligned} (y \Cap z) \to x &= x^* \to (y \Cap z)^* = x^* \to (y^* \Cup z^*) \\ &= (x^* \to y^*) \Cup (x^* \to z^*) = (y \to x) \Cup (z \to x). \end{aligned}$$
$\square$

Proposition 4.5. *The following hold for all* $x, y, z \in \mathcal{Z}(X)$*:*
(1) $(x \Cup y) \to (x \Cup z) \geq_Q x \Cup (y \to z)$;
(2) $(x \Cap y) \to (x \Cap z) \geq_Q x \Cap (y \to z)$.

Proof. (1) Applying Proposition 4.4, since $y \to x \geq_Q x$ we get:

$$\begin{aligned}(x \Cup y) \to (x \Cup z) &= (x \to (x \Cup z)) \Cap (y \to (x \Cup z)) \\ &= ((x \to x) \Cup (x \to z)) \Cap ((y \to x) \Cup (y \to z)) \\ &= (1 \Cup (x \to z)) \Cap ((y \to x) \Cup (y \to z)) \\ &= 1 \Cap ((y \to x) \Cup (y \to z)) \\ &= (y \to x) \Cup (y \to z) \geq_Q x \Cup (y \to z).\end{aligned}$$

(2) Similarly, using Proposition 4.1 we have:

$$\begin{aligned}(x \Cap y) \to (x \Cap z) &= (x \to (x \Cap z)) \Cup (y \to (x \Cap z)) \\ &= ((x \to x) \Cap (x \to z)) \Cup ((y \to x) \Cap (y \to z)) \\ &= (1 \Cap (x \to z)) \Cup ((y \to x) \Cap (y \to z)) \\ &= (x \to z) \Cup ((y \to x) \Cap (y \to z)) \\ &\geq_Q (y \to x) \Cap (y \to z) \geq_Q x \Cap (y \to z).\end{aligned}$$

□

Proposition 4.6. *The following hold for all* $x, y \in \mathcal{Z}(X)$*:*
(1) $(x^* \odot y) \Cap (x \odot y^*) = 0$;
(2) $(x \Cap x^*) \odot (y \Cap y^*) = 0$;
(3) $x \Cap x^* \leq_Q y \Cup y^*$.

Proof. (1) Applying Proposition 2.6(4), we get:

$$\begin{aligned}(x^* \odot y) \Cap (x \odot y^*) &= (x^* \to y^*)^* \Cap (x \to y)^* = (y \to x)^* \Cap (x \to y)^* \\ &= ((y \to x) \Cup (x \to y))^* = 1^* = 0.\end{aligned}$$

(2) By distributivity of $\odot$ over $\Cap$ and using (1), we have:

$$\begin{aligned}(x \Cap x^*) \odot (y \Cap y^*) &= ((x \Cap x^*) \odot y) \Cap ((x \Cap x^*) \odot y^*) \\ &= (x \odot y) \Cap (x^* \odot y) \Cap (x \odot y^*) \Cap (x^* \odot y^*) \\ &= (x \odot y) \Cap 0 \Cap (x^* \odot y^*) = 0.\end{aligned}$$

(3) Using (2), we get:

$$(x \Cap x^*) \to (y \Cup y^*) = ((x \Cap x^*) \odot (y \Cup y^*)^*)^* = ((x \Cap x^*) \odot (y \Cap y^*))^* = 0^* = 1.$$

Since $\leq_Q$ and $\leq$ coincide in $\mathcal{Z}(X)$, it follows that $x \Cap x^* \leq_Q y \Cup y^*$. □

Definition 4.7. A *Kleene algebra* is a structure $(L, \wedge, \vee, ^*, 0, 1)$, where $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and * is a unary operation satisfying the following conditions for all $x, y \in L$:
(K_1) $(x^*)^* = x$;
(K_2) $(x \vee y)^* = x^* \wedge y^*$;
(K_3) $x \wedge x^* \leq y \vee y^*$.

Theorem 4.8. $(\mathcal{Z}(X), \Cap, \Cup, ^*, 0, 1)$ *is a Kleene algebra.*

Proof. It follows from Theorem 4.3 and Proposition 4.6(3). □

5 The OML-center of quantum-Wajsberg algebras

Given a quantum-Wajsberg algebra X, we define the OML-center $\mathcal{O}(X)$ of X, we study its properties, and show that $\mathcal{O}(X)$ is a subalgebra of X. We prove that $\mathcal{O}(X)$ is an orthomodular lattice, and the orthomodular lattices form a subvariety of the variety of quantum-Wajsberg algebras. In what follows, $(X, \rightarrow, ^*, 1)$ will be a quantum-Wajsberg algebra, unless otherwise stated.
Denote $\mathcal{O}(X) = \{x \in X \mid x = x^* \rightarrow x\}$. Obviously $0, 1 \in \mathcal{O}(X)$.

Lemma 5.1. *$\mathcal{O}(X)$ is closed under * and $\rightarrow$.*

Proof. If $x \in \mathcal{O}(X)$, then $x = x^* \rightarrow x$, and by Proposition 2.7(8), we get $x^* = x \rightarrow x^* = (x^*)^* \rightarrow x^*$, hence $x^* \in \mathcal{O}(X)$. Let $x, y \in \mathcal{O}(X)$, that is $x = x^* \rightarrow x$ and $y = y^* \rightarrow y$. By Lemma 2.1(9), we have $(x^* \rightarrow y)^* \rightarrow (x^* \rightarrow y) = (x^* \rightarrow x)^* \rightarrow (y^* \rightarrow y) = x^* \rightarrow y$, thus $x^* \rightarrow y \in \mathcal{O}(X)$. Finally, from $x^*, y \in \mathcal{O}(X)$, we get $x \rightarrow y \in \mathcal{O}(X)$. Hence $\mathcal{O}(X)$ is closed under * and $\rightarrow$. □

Corollary 5.2. *The following hold:*
(1) $\mathcal{O}(X) = \{x \in X \mid x^* = x \rightarrow x^*\}$;
(2) $(\mathcal{O}(X), \rightarrow, 0, 1)$ *is a subalgebra of* $(X, \rightarrow, 0, 1)$;
(3) $\mathcal{O}(X)$ *is closed under* $\Cap$, $\Cup$ *and* $\odot$.

Proposition 5.3. $\mathcal{O}(X) = \{x \in X \mid x^* \Cup x = 1\} = \{x \in X \mid x^* \Cap x = 0\}$.

Proof. If $x \in \mathcal{O}(X)$, then $x = x^* \rightarrow x$, so that $x^* \Cup x = (x^* \rightarrow x) \rightarrow x = x \rightarrow x = 1$. Conversely, if $x^* \Cup x = 1$, then $(x^* \rightarrow x) \rightarrow x = 1$, that is $x^* \rightarrow x \le x$. Since by Proposition 2.6(2), $x \le_Q x^* \rightarrow x$, using Proposition 2.7(4) we get $x = x^* \rightarrow x$, that is $x \in \mathcal{O}(X)$. Similarly $\mathcal{O}(X) = \{x \in X \mid x^* \Cap x = 0\}$. □

Proposition 5.4. *The following hold for all $x \in \mathcal{O}(X)$ and $y \in X$:*
(1) $x \rightarrow (x \rightarrow y) = x \rightarrow y$;
(2) $(x \rightarrow y) \rightarrow x = x$;
(3) $(y \rightarrow x)^* \rightarrow x = y \rightarrow x$;
(4) $(y \rightarrow x)^* \rightarrow (y \rightarrow x) = y \rightarrow (y \rightarrow x)$.

Proof. (1) Using Lemma 2.1(7), we get: $x \to (x \to y) = (x \to y)^* \to x^* = x \to (y^* \to x^*) = y^* \to (x \to x^*) = y^* \to x^* = x \to y$.
(2) It follows by (1), applying Proposition 2.7(8).
(3) By Lemma 2.1(7), $(y \to x)^* \to x = y \to (x^* \to x) = y \to x$.
(4) Replacing y by y^* in Lemma 2.1(9) and taking into consideration that $x^* \to x = x$, we get $(x^* \to y^*)^* \to (x^* \to y^*) = x^* \to (y \to y^*)$, so that $(y \to x)^* \to (y \to x) = y \to (x^* \to y^*)$. Hence $(y \to x)^* \to (y \to x) = y \to (y \to x)$. □

For any $x, y \in \mathcal{O}(X)$, define the operations: $x \Cup_L y = x^* \to y$, $x \Cap_L y = x \odot y$ and the relation $x \leq_L y$ iff $x^* \to y = y$. One can easily check that $x \Cup_L y = (x^* \Cap_L y^*)^*$ and $x \Cap_L y = (x^* \Cup_L y^*)^*$.

Proposition 5.5. *The following hold for all $x, y \in \mathcal{O}(X)$:*
(1) $\leq_L = \leq_{Q|\mathcal{O}(X)}$;
(2) $x \Cup y \leq_Q x \Cup_L y$ *and* $x \Cap_L y \leq_Q x \Cap y$;
(3) $(x \Cup_L y) \to x^* = x^*$ *and* $(x \Cap_L y)^* \to x = x$;
(4) $(x \Cup_L y)^* \to y = x \Cup_L y$ *and* $(x \Cap_L y) \to y^* = (x \Cap_L y)^*$;
(5) $(\mathcal{O}(X), \Cap_L, \Cup_L, 0, 1)$ *is a bounded lattice.*

Proof. (1) Let $x, y \in \mathcal{O}(X)$ such that $x \leq_Q y$. It follows that $y^* \leq_Q x^*$ and $x^* \to y \leq_Q y^* \to y = y$. On the other hand, $y \leq_Q x^* \to y$, hence $x^* \to y = y$, that is $x \leq_L y$. Conversely, if $x \leq_L y$ we have $x \leq_Q x^* \to y = y$. Thus $\leq_L = \leq_{Q|\mathcal{O}(X)}$.
(2) Since $x^* \leq_Q x \to y$, we have $(x \to y) \to y \leq_Q x^* \to y$, that is $x \Cup y \leq_Q x \Cup_L y$. Similarly $x \leq_Q x^* \to y^*$ implies $(x^* \to y^*) \to y^* \leq_Q x \to y^*$. Hence $(x \to y^*)^* \leq_Q x \Cap y$, so that $x \odot y \leq_Q x \Cap y$, that is $x \Cap_L y \leq_Q x \Cap y$.
(3) It follows from Proposition 5.4(2), replacing x by x^* and y by y^*, respectively.
(4) Since $y \in \mathcal{O}(X)$, by Proposition 5.4(3) we have $(x \to y)^* \to y = x \to y$. Replacing x by x^* we get $(x \Cup_L y)^* \to y = x \Cup_L y$, and replacing y by y^* we have $(x \Cap_L y) \to y^* = (x \Cap_L y)^*$.
(5) Clearly $\Cup_L$ and $\Cap_L$ are commutative and idempotent. Moreover, using Lemma 2.1(7) we can easily check that $\Cup_L$ and $\Cap_L$ are associative. Finally, applying Proposition 5.4(2), we have:

$$x \Cup_L (x \Cap_L y) = x^* \to (x \to y^*)^* = (x \to y^*) \to x = x,$$
$$x \Cap_L (x \Cup_L y) = (x \to (x^* \to y)^*)^* = ((x^* \to y) \to x^*)^* = (x^*)^* = x,$$

for all $x, y \in \mathcal{O}(X)$, hence $\Cup_L$ and $\Cap_L$ satisfy the absorption laws.
Thus $(\mathcal{O}(X), \Cap_L, \Cup_L, 0, 1)$ is a bounded lattice. □

Corollary 5.6. *The following hold for all $x, y \in \mathcal{O}(X)$:*
(1) $x \leq_Q y$ *iff* $y = y \Cup_L x$;
(2) $x \Cup y = (x \to y)^* \Cup_L y$.

Proof. (1) $x \leq_Q y$ iff $x \leq_L y$ iff $y = x^* \to y = y^* \to x = y \Cup_L x$.
(2) $x \Cup y = (x \to y) \to y = ((x \to y)^*)^* \to y = (x \to y)^* \Cup_L y$. □

In what follows, if $x, y \in \mathcal{O}(X)$, we will use $x \leq_Q y$ instead of $x \leq_L y$.

Proposition 5.7. *For any $x, y \in \mathcal{O}(X)$, $x \Cup_L y$ and $x \Cap_L y$ are the l.u.b. and g.l.b. of $\{x, y\}$, respectively.*

Proof. Obviously $x, y \leq_Q x^* \to y$, so that $x \Cup_L y$ is an upper bound of $\{x, y\}$. Let $z \in \mathcal{O}(X)$ be another upper bound of $\{x, y\}$ in $\mathcal{O}(X)$, that is $x, y \leq_Q z$. It follows that $z^* \leq_Q x^*$, so that $x^* \to y \leq_Q x^* \to z \leq_Q z^* \to z = z$. Hence $x \Cup_L y \leq_Q z$, that is $x \Cup_L y$ is the l.u.b. of $\{x, y\}$. Similarly $x \odot y \leq_Q x, y$, thus $x \Cap_L y$ is a lower bound of $\{x, y\}$. Let $z \in \mathcal{O}(X)$ be another lower bound of $\{x, y\}$ in $\mathcal{O}(X)$, so that $z \leq_Q x, y$. We get $x^*, y^* \leq_Q z^*$, so that z^* is an upper bound of $\{x^*, y^*\}$, hence $x^* \Cup_L y^* \leq_Q z^*$, that is $x \to y^* \leq_Q z^*$. It follows that $z \leq_Q (x \to y^*)^* = x \odot y = x \Cap_L y$, thus $x \Cap_L y$ is the g.l.b. of $\{x, y\}$. □

Definition 5.8. (*[2]*) An algebra $(X, \wedge, \vee, ^{'}, 0, 1)$ with two binary, one unary and two nullary operations is an *ortholattice* if it satisfies the following axioms for all $x, y, z \in X$:
(Q_1) $(X, \wedge, \vee, 0, 1)$ is a bounded lattice;
(Q_2) $x \wedge x^{'} = 0$ and $x \vee x^{'} = 1$;
(Q_3) $(x \wedge y)^{'} = x^{'} \vee y^{'}$ and $(x \vee y)^{'} = x^{'} \wedge y^{'}$;
(Q_4) $(x^{'})^{'} = x$.
An *orthomodular lattice* is an ortholattice satisfying the following axiom:
(Q_5) $x \leq y$ implies $x \vee (x^{'} \wedge y) = y$ (where $x \leq y$ iff $x = x \wedge y$).

Theorem 5.9. $(\mathcal{O}(X), \Cap_L, \Cup_L, ^*, 0, 1)$ is an orthomodular lattice called the *orthomodular center* or *OML-center of X*.

Proof. Let X be a quantum-Wajsberg algebra. Using Propositions 5.5, 5.7, 5.3 we can easily check that $(\mathcal{O}(X), \Cap_L, \Cup_L, ^*, 0, 1)$ is an ortholattice. We show that axiom (Q_5) is also satisfied. Let $x, y \in \mathcal{O}(X)$ such that $x \leq_Q y$, and we have: $x \Cup_L (x^* \Cap_L y) = x \Cup_L (x^* \odot y) = x \Cup_L (x^* \to y^*)^* = x \Cup_L (y \to x)^* = x^* \to (y \to x)^* = (y \to x) \to x = y \Cup x = y$, since $x \leq_Q y$. □

Theorem 5.10. *If $(X, \wedge, \vee, ^{'}, 0, 1)$ is an orthomodular lattice, then $(X, \to, 0, 1)$ is a quantum-Wajsberg algebra, where $x \to y = x^{'} \vee y$ for all $x, y \in X$.*

Proof. According to [5, Thm. 2.3.9], every orthomodular lattice $(X, \wedge, \vee, ^{'}, 0, 1)$ determines a QMV algebra by taking $\oplus$ as the supremum $\vee$ and * as the orthocomplement $^{'}$, and conversely, if an ortholattice X determines a QMV algebra

$(X, \oplus, ^*, 0, 1)$ taking $\oplus = \vee$ and $^* = '$, then X is orthomodular. By [4, Thm. 5.3], any quantum-MV algebra $(X, \oplus, ^*, 0, 1)$ is a quantum-Wajsberg algebra $(X, \rightarrow, 0, 1)$, where $x \rightarrow y = x^* \oplus y$. It follows that every orthomodular lattice $(X, \wedge, \vee, ', 0, 1)$ determines a quantum-Wajsberg algebra $(X, \rightarrow, 0, 1)$ with $x \rightarrow y = x^* \oplus y = x^* \vee y$ for all $x, y \in X$. □

Corollary 5.11. *$(X, \Cap_L, \Cup_L, ^*, 0, 1)$ is an orthomodular lattice if and only if $\mathcal{O}(X) = X$.*

Similarly as [5, Cor. 2.3.13] for the case of QMV algebras, we have the following result.

Corollary 5.12. *The orthomodular lattices form a subvariety of the variety of quantum-Wajsberg algebras. This subvariety satisfies the condition $x = x^* \rightarrow x$, or equivalently, $x^* \Cup x = 1$, or equivalently, $x^* \Cap x = 0$.*

Proof. The equivalence of conditions $x = x^* \rightarrow x$, $x^* \Cup x = 1$ and $x^* \Cap x = 0$ follows from Proposition 5.3. If a quantum-Wajsberg algebra $(X, \rightarrow, 0, 1)$ is an orthomodular lattice with $x \vee y = x^* \rightarrow y$, than $x^* \rightarrow x = x \vee x = x$. Conversely, if X satisfies condition $x^* \rightarrow x = x$ for any $x \in X$, then $\mathcal{O}(X) = X$, hence X is an orthomodular lattice. □

Example 5.13. *Let $X = \{0, a, b, c, d, 1\}$ and let $(X, \rightarrow, 0, 1)$ be the involutive BE algebra with $\rightarrow$ and the corresponding operation $\Cap$ given in the following tables:*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	c	1	1
b	d	1	1	1	d	1
c	a	a	1	1	1	1
d	b	1	b	1	1	1
1	0	a	b	c	d	1

$\Cap$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	0	d	a
b	0	a	b	c	0	b
c	0	0	b	c	d	c
d	0	a	0	c	d	d
1	0	a	b	c	d	1

Then X is a quantum-Wajsberg algebra and $\mathcal{Z}(X) = \{0, 1\}$, $\mathcal{O}(X) = X$. Therefore $(X, \Cap_L, \Cup_L, ^*, 0, 1)$ is an orthomodular lattice with $\Cup_L$ and $\Cap_L$ given below.

$\Cup_L$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	1	1
b	b	1	b	1	1	1
c	c	1	1	c	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

$\Cap_L$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	b	0	0	b
c	0	0	0	c	0	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

As we can see in this example, in general, $\Cup_L \neq \Cup$ *and* $\Cap_L \neq \Cap$.

Remark 5.14. In general, the lattice $(\mathcal{O}(X), \Cap_L, \Cup_L, 0, 1)$ is not distributive. Indeed, in Example 5.13 we have $a \Cup_L (b \Cap_L c) = a \neq 1 = (a \Cup_L b) \Cap_L (a \Cup_L c)$.

6 Concluding remarks and future work

In this paper, we continued the study of quantum-Wajsberg algebras ([4]). We defined the Wajsberg-center and the OML-center of a quantum-Wajsberg algebra $(X, \rightarrow, ^*, 1)$, proving that the Wajsberg-center is a Wajsberg subalgebra of X, and that it is a distributive sublattice of the poset $(X, \leq_Q, 0, 1)$ (where $0 = 1^*$). We introduced the notion of quasi-linear quantum-Wajsberg algebras, and we proved that the Wajsberg-center of a quasi-linear quantum-Wajsberg algebra is a linearly ordered Wajsberg algebra. We also proved that the OML-center is an orthomodular lattice, and that the orthomodular lattices form a subvariety of the variety of quantum-Wajsberg algebras.
There are several ways this work can be continued, as follows:

– Introduce and study certain generalizations of quantum-Wajsberg algebras, such as implicative-orthomodular, pre-Wajsberg and meta-Wajsberg algebras.

– Define the implicative-orthomodular lattices as a special subclass of quantum-Wajsberg algebras, and study their properties.

– Prove an analogue of Foulis-Holland theorem for implicative-orthomodular lattices.

– Study the Baer *-semigroup associated to an implicative-orthomodular lattice X and its relationship with the Sasaki projections defined on X.

– Investigate the central lifting property for implicative-orthomodular lattices.

Another direction of research could be the solving of the following open problem.
Open problem. Is the variety of quasi-linear quantum-Wajsberg algebras axiomatizable (in the sense of [9])?

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Flexible Involutive Meadows

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Abstract

We investigate a notion of inverse for neutrices inspired by Van den Berg and Koudjeti's decomposition of a neutrix as the product of a real number and an idempotent neutrix. We end up with an algebraic structure that can be characterized axiomatically and generalizes involutive meadows. The latter are algebraic structures where the inverse for multiplication is a total operation. As it turns out, the structures satisfying the axioms of flexible involutive meadows are of interest beyond nonstandard analysis.

1 Introduction

Neutrices and external numbers (which can be seen as translations of neutrices over the hyperreal line) were introduced by Van den Berg and Koudjeti in [25] as models of uncertainties, in the context of nonstandard analysis, and further developed in [24, 27, 16, 19, 20]. Neutrices were named after and inspired by Van der Corput's groups of functions [28] in an attempt to give a mathematically rigorous formulation to the art of neglecting small quantities – *ars negligendi*.

One of the long-standing open questions in the theory of external numbers is the definition of a suitable notion of inverse of a neutrix. For zeroless external numbers,

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that is, external numbers that don't contain 0 and therefore cannot be reduced to a neutrix, there is a sort of inverse, defined from the Minkowski product between sets (see Definition 3.4 below), but this cannot work as a proper inverse since in many instances the result turns out to be the empty set.

A meadow (see Sections 2.1 and 2.2 below for further details) is a sort of commutative ring with a multiplicative identity element and a total multiplicative inverse operation. The theory of meadows allows for two main options: (*i*) involutive meadows which define $0^{-1} = 0$, resulting into an equational theory closer to that of the original structure [6], (*ii*) common meadows which define 0^{-1} as a new error term that propagates through calculations [5] (see also [7, 15, 8, 9]). For some recent developments, see [8, 9, 12, 13, 14].

One of the motivations for the study of structures where the inverse of zero is defined comes from equational theories [23, 26, 6]. For instance, Ono and Komori introduced such structures motivated from the algebraic study of equational theories and universal theories of fields, and free algebras over all fields, respectively. A long-standing result by Birkhoff states that algebraic structures with an equational axiomatization – namely, whose axioms only involve equality, besides the functions and constants of the structure itself – are closed under substructures. Algebraic structures where the inverse is defined only for nonzero elements are not equational, since they have to use inequalities or quantifiers in their definition of a multiplicative inverse. Instead, involutive meadows and common meadows which, as mentioned above, define the inverse of zero as zero or a new error term, respectively, admit equational axiomatizations.

Equational axiomatizations of meadows based on known algebraic structures, such as $\mathbb{Q}$ and $\mathbb{R}$, are also of interest to computer science. According to Bergstra and Tucker [6], such equational axiomatizations allow for simple term rewriting systems and are easier to automate in formal reasoning.

Another motivation for the study of meadows is a philosophical interest in the definition of an inverse of zero (see e.g. [4, Section 3]), if one wants to assign a meaning to expressions such as 0^{-1} or $1/0$ (Bergstra and Middleburg argue that, in principle, these two operations need to be distinguished [4]).

It turns out that external numbers are particularly suitable for expressing the kind of concepts involved in the definition of the inverse of zero. The key insight is that, being convex subgroups of the hyperreal numbers (i.e. the extension of the real number system which includes nonstandard elements such as infinitesimals), neutrices are "error" terms in the following sense:

- the sum of a neutrix with itself or with one of its elements is still the same neutrix;

- the product of a neutrix by an appreciable (not infinitesimal and not infinitely large) number is still the same neutrix;
- the product of a neutrix by an external number is a neutrix.

These properties are similar to those of 0, that is neutrices are idempotent for addition and absorbent for multiplication. Therefore neutrices can be seen as generalized zeroes and are suitable to build models of meadows.

The fact that one is using hyperreals (or other non-archimedean field extensions of the real numbers) is crucial, because the real numbers only have two convex subgroups: $\{0\}$ and $\mathbb{R}$, while in the context of the hyperreals there are countably infinitely many, e.g. the set of all infinitesimals – denoted $\oslash$ – and the set of all limited numbers – denoted $\pounds$ (see the examples after Definition 3.1). In turn, external numbers are of the form $a+A$, where a is an hyperreal number and A is a neutrix and can therefore be seen as translations of neutrices. According to the interpretation of neutrices as error terms or generalized zeroes, external numbers can be interpreted as expressing a quantity with a degree of uncertainty.

By introducing an alternative way to define the inverse of a neutrix, inspired by a result of Van den Berg and Koudjeti [25] stating that every neutrix can be decomposed as the product of an hyperreal number and an idempotent neutrix, we end up with an algebraic structure that can be characterized axiomatically and generalizes involutive meadows. Since the new class of structures involves error terms, we call it the class of *flexible involutive meadow*, in the spirit of [22].

In summary, the contributions of the paper are the following.

- We answer a question about the inverse of a neutrix.
- We connect the inverse of a neutrix to meadows, specifically involutive meadows.
- Inspired by the properties of the inverse of a neutrix, we propose a new algebraic structure that generalises involutive meadows in a simple way, called flexible involutive meadows, and we provide an axiomatization of such structures.
- We give some models of flexible involutive meadows.

We start in Section 2 by recalling the axioms of common meadows and of involutive meadows. We also recall some notions and results concerning external numbers in Section 3. In Section 4 we introduce flexible involutive meadows and prove that the external numbers are a flexible involutive meadow. We also derive some properties of flexible involutive meadows and relate them with varieties and von Neumann regular rings. Then, in Section 5, we present additional models for meadows relying on the external numbers. Some final remarks and open questions are mentioned in Section 6.

2 Preliminary notions

In this section we recall the axioms of common meadows and involutive meadows. We also provide some motivation for the study of structures where division by zero is possible.

2.1 Involutive meadows

The axioms of involutive meadows are listed in Figure 1 (see also [2, 4]).

(I_1)	$(x+y)+z = x+(y+z)$
(I_2)	$x+y = y+x$
(I_3)	$x+0 = x$
(I_4)	$x+(-x) = 0$
(I_5)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(I_6)	$x \cdot y = y \cdot x$
(I_7)	$1 \cdot x = x$
(I_8)	$x \cdot (y+z) = x \cdot y + x \cdot z$
(I_9)	$(x^{-1})^{-1} = x$
(I_{10})	$x \cdot (x \cdot x^{-1}) = x$

Figure 1: Axioms for involutive meadows

The term *involutive* refers to the fact that taking inverses is an involution, as postulated by axiom (I_9). With the exception of axiom (I_{10}), the remaining axioms are quite standard, as they postulate the existence of operations of addition $+$ and multiplication $\cdot$ which are associative, commutative, admit a neutral element (denoted 0 and 1 respectively). Furthermore, there is an inverse for addition, and multiplication is distributive with respect to addition. Axioms (I_9) and (I_{10}) entail that $0^{-1} = 0$ (see [6, Theorem 2.2]). Axiom (I_{10}) replaces the more usual $x \cdot x^{-1} = 1$, which is false for $x = 0$ (otherwise, $0 = 0 \cdot 0 = 0 \cdot 0^{-1} = 1$). This hints at the fact that, in general, one should not define x^{-1} as the element satisfying $x \cdot x^{-1} = 1$. This also ties in with rejecting division as the "inverse" of multiplication, as discussed in [4].

2.2 Common meadows

The axioms of common meadows are listed in Figure 2 (see also [5]).

(M_1)	$(x + y) + z = x + (y + z)$
(M_2)	$x + y = y + x$
(M_3)	$x + 0 = x$
(M_4)	$x + (-x) = 0 \cdot x$
(M_5)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(M_6)	$x \cdot y = y \cdot x$
(M_7)	$1 \cdot x = x$
(M_8)	$x \cdot (y + z) = x \cdot y + x \cdot z$
(M_9)	$-(-x) = x$
(M_{10})	$x \cdot x^{-1} = 1 + 0 \cdot x^{-1}$
(M_{11})	$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$
(M_{12})	$(1 + 0 \cdot x)^{-1} = 1 + 0 \cdot x$
(M_{13})	$0^{-1} = \mathbf{a}$
(M_{14})	$x + \mathbf{a} = \mathbf{a}$

Figure 2: Axioms for common meadows

As with involutive meadows, some of the axioms are quite standard (namely (M_1) – (M_3), (M_5) – (M_7), (M_8), (M_9), and (M_{11})), as they postulate the existence of operations of addition + and multiplication · which are associative, commutative and admit a neutral element (denoted 0 and 1 respectively). Note that, in involutive meadows, the equations of axioms (M_9) and (M_{11}) can be derived from the other axioms (as discussed in [6]). Furthermore, there is an inverse for addition, multiplication is distributive with respect to addition, and the inverse of the product of two elements is the product of the inverses.

Axiom (M_4) postulates the existence of a sort of additive inverse for every element

x but with the caveat that the result of operating an element with its inverse is not the neutral element 0 but $0 \cdot x$.

Axioms (M_{10}) and (M_{12}) concern further properties of the inverse for multiplication. The novelty, compared with more familiar settings, is that they have "error" terms in the form of the product of an element x, (respectively, its inverse x^{-1}) by 0.

Axiom (M_{13}) defines 0^{-1} as an "error" term $\mathbf{a}$ (some authors denote this error term by $\perp$) that does not belong to the initial structure. Due to the presence of this error term, the result of $x \cdot x^{-1}$ is defined as $1 + 0 \cdot x^{-1}$. If $x \neq 0$ and $x \neq \mathbf{a}$, then $0 \cdot x^{-1} = 0$ (see [5, Proposition 2.3.1]) and we recover the usual result that holds in a field. If $x = 0$ or $x = \mathbf{a}$, then the additional term $0 \cdot x^{-1}$ is equal to $\mathbf{a}$.

Axiom (M_{12}) has a similar motivation: if $x \neq \mathbf{a}$, then we recover that the inverse of 1 is 1. If $x = \mathbf{a}$, then we get that the inverse of $\mathbf{a}$ is $\mathbf{a}$ itself.

3 Hyperreal numbers and external numbers

Let us recall some definitions and results about neutrices and external numbers. We will use ${}^*\mathbb{R}$ to denote an elementary equivalent extension of the real number system that includes nonstandard elements – such as infinitesimals – [21], and $\mathbb{R}$ to denote the usual set of real numbers.[1]

A number x is *infinitesimal* if $|x| < r$ for every positive $r \in \mathbb{R}$ and it is *infinite* if $|x| > r$ for every $r \in \mathbb{R}$. We use the notation $x \simeq 0$ to say that x is infinitesimal. We will also write $x \simeq y$, and say that x is *infinitely close* to y, if $x - y \simeq 0$. A number is said to be *finite* if it is not infinite, and *appreciable* if it is neither infinitesimal nor infinite.

Crucially, the hyperreals admit nontrivial convex subgroups for addition (for instance: the set of infinitesimals $\oslash$, the set of finite numbers $\pounds$).

This property is shared by other non-archimedean field extensions of $\mathbb{R}$. We will discuss this matter further in Section 4.4.

3.1 External numbers

Definition 3.1 (Neutrices)**.** A *neutrix* is a convex subset of ${}^*\mathbb{R}$ that is a subgroup for addition.

Some simple examples of neutrices are:

- $\oslash = \{x \in {}^*\mathbb{R} : x \simeq 0\}$;

[1] Note that this notation differs from the usual presentations of external numbers, according to which $\mathbb{R}$ already contains nonstandard elements.

- $\pounds = \{x \in {}^*\mathbb{R} : x \text{ is finite}\}$;
- if $\varepsilon \simeq 0$, $\varepsilon\pounds = \{x \in {}^*\mathbb{R} : \frac{x}{\varepsilon} \text{ is finite}\}$.

Definition 3.2 (External numbers)**.** An *external number* α is the sum of an hyperreal number a and a neutrix A in the following sense:

$$\alpha = a + A = \{a + r : r \in A\}.$$

An external number $\alpha = a + A$ that is not reduced to a neutrix (equivalently, such that $0 \notin \alpha$) is said to be *zeroless*.

The sum and product of external numbers is introduced in the following definition. We refer to [25, 16, 20] for their properties.

Definition 3.3. For $a, b \in {}^*\mathbb{R}$ and $A, B \subseteq {}^*\mathbb{R}$ (not necessarily neutrices), we define the Minkowski sum and product

$$(a + A) + (b + B) = (a + b) + (A + B)$$
$$(a + A) \cdot (b + B) = ab + aB + bA + AB,$$

where

$$A + B = \{x + y : x \in A \wedge y \in B\}$$
$$aB = \{ay : y \in B\}$$
$$AB = \{xy : x \in A \wedge y \in B\}.$$

It is also possible to define a notion of division between subsets of hyperreal numbers, even if they contain 0.

Definition 3.4. For $A, B \subseteq {}^*\mathbb{R}$ (not necessarily neutrices), we define

$$A \cdot B^{-1} = \{x : xB \subseteq A\}.$$

Usually, for neutrices, $A \cdot B^{-1}$ is written as $\frac{A}{B}$. Here we chose the inverse notation since we are investigating structures related to meadows, whose axioms are commonly stated in terms of the inverse operation. For further discussion on the use of these operations we refer to [4].

Notice that Definition 3.4 doesn't allow us to obtain a *proper* inverse of a neutrix. In fact, if $A = \{1\}$ and B is a neutrix, $A \cdot B^{-1}$ is empty, since for no x we have $0 \cdot x \in A$. This example motivated us to look for alternative definitions of inverses of a neutrix.

Let us finish this section by recalling some results on external numbers which will be useful later on.

If $x = a + A$ is zeroless, then $a^{-1} \cdot A \subseteq A$, and therefore $a^{-1} \cdot A + A = A$. In fact, $a^{-1} \cdot A \subseteq \oslash$.

The Taylor formula for $(a + A)^{-1}$ allows to obtain the following expression for the inverse of zeroless external numbers.

Proposition 3.5 ([25, p. 151],[20, Theorem 1.4.2])**.** *Let $x = a + A$ be a zeroless external number. Then*

$$x^{-1} = (a + A)^{-1} = a^{-1} + a^{-2} \cdot A.$$

We conclude with a list of basic algebraic properties of external numbers. Properties (3) and (4) are a consequence of the fact that, for a neutrix A, $A+A = A-A = A$. Property (7) replaces the usual distributivity formula, taking into account how error terms propagate (for further details on the distributivity formula for external numbers, see [16, Section 5]).

Proposition 3.6. *Let x, y, z be external numbers such that $x = a + A$. Then*

1. $x + (y + z) = (x + y) + z$;
2. $x + y = y + x$;
3. $x + A = x$;
4. $x + (-x) = A$;
5. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
6. $x \cdot y = y \cdot x$;
7. $x \cdot y + x \cdot z = x \cdot (y + z) + A \cdot y + A \cdot z$;
8. $(x^{-1})^{-1} = x$.

4 Flexible involutive meadows

A neutrix I is said to be *idempotent* if $I \cdot I = I$. As showed by Van den Berg and Koudjeti in [25] (see also [17]) every neutrix is a multiple of an idempotent neutrix.

Theorem 4.1 ([25, Theorem 7.4.2])**.** *Let N be a neutrix. Then, there exists an hyperreal number r and a unique idempotent neutrix I such that $N = r \cdot I$.*

We use the previous result to define inverses for neutrices.

Definition 4.2. Let $x = a + A$, where $A = r \cdot I$, for some hyperreal number r and idempotent neutrix I. We define the *inverse* of x, denoted x^{-1}, as follows:

$$x^{-1} = \begin{cases} a^{-1} + a^{-2} \cdot A & \text{if } a \neq 0 \\ r^{-1} \cdot I & \text{otherwise.} \end{cases}$$

The idempotent neutrices I can be seen as a generalized zeroes, since they share with 0 the properties $I + I = I$ and $I \cdot I = I$. Also, $I^{-1} = I$, similarly to $0^{-1} = 0$ in involutive meadows.

In the decomposition $N = r \cdot I$ of Theorem 4.1, the idempotent neutrix I is uniquely determined, but the number r is not. Nevertheless, the inverse given in Definition 4.2 is uniquely defined, as a consequence of the next proposition.

Proposition 4.3. *If $r, s \in {}^*\mathbb{R}$ and $r \neq s$ satisfy $N = r \cdot I = s \cdot I$, then also $r^{-1} \cdot I = s^{-1} \cdot I$.*

Proof. We may assume, without loss of generality that $0 < r < s$, which implies that $s^{-1} < r^{-1}$. Suppose towards a contradiction that $r^{-1} \cdot I \neq s^{-1} \cdot I$. By our assumptions over r and s, this implies $s^{-1} \cdot I \subsetneq r^{-1} \cdot I$. Then, there exists some $i \in I$ such that $i \cdot r^{-1} \notin s^{-1} \cdot I$. If we multiply by r, we obtain

$$i \notin s^{-1} \cdot (r \cdot I) = s^{-1} \cdot (s \cdot I) = I,$$

which contradicts the assumption that $i \in I$. Hence $r^{-1} \cdot I = s^{-1} \cdot I$. □

We show that the external numbers equipped with the inverse defined in Definition 4.2 satisfy the axioms given in Figure 3, where $N(x)$ denotes the neutrix part of the external number x. As such, one can also think of $N(x)$ as an error term, or a generalized zero, such that every x decomposes uniquely as $x = r + N(x)$ with $N(r) = 0$. We call any structure satisfying the axioms in Figure 3 a *flexible involutive meadow.*

(FI_1)	$(x+y)+z = x+(y+z)$
(FI_2)	$x+y = y+x$
(FI_3)	$x+N(x) = x$
(FI_4)	$x+(-x) = N(x)$
(FI_5)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(FI_6)	$x \cdot y = y \cdot x$
(FI_7)	$(1+N(x) \cdot x^{-1}) \cdot x = x$
(FI_8)	$x \cdot y + x \cdot z = x \cdot (y+z) + N(x) \cdot y + N(x) \cdot z$
(FI_9)	$(x^{-1})^{-1} = x$
(FI_{10})	$x \cdot (x \cdot x^{-1}) = x$

Figure 3: Axioms for flexible involutive meadows

The axioms of flexible involutive meadows generalize the axioms of involutive meadows given in Figure 1 by replacing 0 with generalized zeroes $N(x)$, 1 with generalized ones (i.e. 1 plus an error term of the form $N(x)$), and distributivity by a generalized form of distributivity which holds for the external numbers. Notice that, in the context of the external numbers, the generalized zeroes take the form of neutrices.

Axiom (FI_7) is the flexible counterpart to (I_7), but replaces 1 with $1+N(x) \cdot x^{-1}$, and not simply with $1+N(x)$. In the setting of external numbers, this is necessary because if $N(x) \supseteq \pounds$, then $1+N(x) = N(x)$ would be a generalized zero. Axiom (FI_8) is a generalized distributivity axiom. In the setting of external numbers, the term $N(x) \cdot y + N(x) \cdot z$ is a neutrix, so the error term in (FI_8) is once again a neutrix. In fact, if one interprets $N(x)$ as being 0 for all x, then one recovers the axioms for involutive meadows. The proof is straightforward.

Lemma 4.4. *Let M be an involutive meadow. Then, if one defines $N(x) = 0$ for all $x \in M$, the resulting structure is a flexible involutive meadow.*

In order to prove that the external numbers satisfy the axioms for flexible involutive meadows, we will use the following properties of the inverse of a neutrix.

Lemma 4.5. *Let $x = N(x) = r \cdot I$, with $r \in {}^*\mathbb{R}$ and I an idempotent neutrix. Then:*

1. $(1 + x \cdot x^{-1}) \cdot x = x$;

2. $(x^{-1})^{-1} = x$;

3. $x \cdot (x \cdot x^{-1}) = x$.

Proof. 1. We have that $x \cdot x^{-1} = N(x) \cdot x^{-1} = I$. Hence

$$(1 + x \cdot x^{-1}) \cdot x = (1 + I) \cdot (r \cdot I) = r \cdot I + r \cdot I^2 = r \cdot I = x.$$

2. We have
$$(x^{-1})^{-1} = (r^{-1}I)^{-1} = (r^{-1})^{-1}I = rI = x.$$

3. We have

$$x \cdot (x \cdot x^{-1}) = rI \cdot (rI \cdot r^{-1}I) = rI \cdot I = rI = x. \qquad \square$$

As proved below, the external numbers satisfy an additional property related to the following *Inverse Law* of involutive meadows:

$$x \neq 0 \Rightarrow x \cdot x^{-1} = 1$$

Involutive meadows that satisfy the Inverse Law are called *cancellation meadows* and are of particular interest. In fact, in [1] it is proved that every involutive meadow is a subdirect product of cancellation meadows.

In the setting of flexible involutive meadows, the inverse law is more suitably expressed by its flexible counterpart:

$$x \neq N(x) \Rightarrow x \cdot x^{-1} = 1 + e, \tag{4.1}$$

where e is a generalized zero (in the sense that $e + e = e$) such that $1 + e$ is not a generalized zero.

Moreover, by part 3 and part 4 of Proposition 3.6, the external numbers satisfy the following properties that generalize the properties of arithmetical meadows [4]:

$$\begin{array}{ll} (\mathrm{A}_1) & x + (-x) = N(x) \\ (\mathrm{A}_2) & x + N(x) = x. \end{array}$$

Notice that (A_2) is axiom (FI_3) of flexible involutive meadows.

Theorem 4.6. *The external numbers with the usual addition and multiplication and with the inverse introduced in Definition 4.2 satisfy the axioms for flexible involutive meadows plus the Flexible Inverse Law given by* (4.1) *and the properties* (A_1) *and* (A_2)*.*

Proof. By Proposition 3.6, in order to show that the external numbers are a flexible involutive meadow we only need to verify (FI_7) and (FI_{10}). If x is a neutrix, both axioms hold due to Lemma 4.5. Assume that $x = a + A$ is zeroless. Then, using the algebraic properties of the external numbers one derives

$$\begin{aligned}(1 + N(x) \cdot x^{-1}) \cdot x &= \left(1 + A\left(a^{-1} + a^{-2} \cdot A\right)\right)(a + A)\\ &= \left(1 + \left(a^{-1} \cdot A + a^{-2} \cdot A^2\right)\right)(a + A)\\ &= a + A + a^{-1} \cdot A^2 + A + a^{-1} \cdot A^3\\ &= a + A = x.\end{aligned}$$

Hence (FI_7) holds. As regarding (FI_{10}) one has

$$x(x \cdot x^{-1}) = (a + A)\left(1 + a^{-1} \cdot A\right) = a + A + A + a^{-1} \cdot A^2 = a + A = x.$$

Hence (FI_{10}) also holds and therefore the external numbers are a flexible involutive meadow.

We now show the Flexible Inverse Law. Let $x = a + A$ be a zeroless external number. Then

$$x \cdot x^{-1} = 1 + a^{-1} \cdot A + a^{-1} \cdot A + a^{-2} \cdot A^2 = 1 + a^{-1} \cdot A.$$

Since x is zeroless, $a^{-1} \cdot A \subseteq \oslash$, so the Flexible Inverse Law is satisfied. □

Corollary 4.7. *The axioms for flexible involutive meadows are consistent.*

4.1 Some properties of flexible involutive meadows

We now prove some basic properties of flexible involutive meadows. We start by showing an additive cancellation law and that $N(\cdot)$ is idempotent for addition.

Proposition 4.8. *Let M be a flexible involutive meadow and let $x, y, z \in M$. Then*

1. *$x + y = x + z$ if and only if $N(x) + y = N(x) + z$;*
2. *$N(x) + N(x) = N(x)$.*

Proof. 1. Suppose firstly that $x + y = x + z$. Then

$$N(x) + y = -x + x + y = -x + x + z = N(x) + z.$$

Suppose secondly that $N(x) + y = N(x) + z$. Then

$$x + y = x + N(x) + y = x + N(x) + z = x + z.$$

2. This follows from applying part (1) to axiom (FI_3). □

In order to prove other basic properties that allow one to operate with the $N(\cdot)$ function and with additive inverses one needs to assume the following two extra axioms

$$\begin{array}{ll}(N_1) & N(x+y)=N(x)\vee N(x+y)=N(y);\\ (N_2) & N(-x)=N(x).\end{array}$$

Proposition 4.9. *Let M be a flexible involutive meadow satisfying also* (N_1) *and* (N_2)*, and let $x, y, z \in M$. Then*

1. $N(x+y)=N(x)+N(y)$*;*
2. $N(N(x))=N(x)$*;*
3. *If $x=N(y)$, then $x=N(x)$;*
4. $-(-x)=x$*;*
5. $-(x+y)=-x-y$*;*
6. $N(x)=-N(x)$*.*

Proof. 1. One has

$$x+y+N(x)+N(y)=x+N(x)+y+N(y)=x+y.$$

Then by part (1) of Proposition 4.8

$$N(x+y)+N(x)+N(y)=N(x+y). \tag{4.2}$$

By (N_1) one has $N(x+y)=N(x)$ or $N(x+y)=N(y)$. Suppose that $N(x+y)=N(x)$. Then by (4.2) and Proposition 4.8,

$$N(x+y)=N(x)+N(x)+N(y)=N(x)+N(y).$$

If $N(x+y)=N(y)$ the proof is analogous.

2. Using Proposition 4.8 and part 1 we have

$$N(N(x))=N(x-x)=N(x)+N(-x)=N(x)+N(x)=N(x).$$

3. Using part 2 we derive that $N(x) = N(N(y)) = N(y) = x$.

4. We have that
$$N(-(-x)) = N(-x) = N(x) = -x + x.$$
Hence
$$-(-x) = -(-x) + N(-(-x)) = -(-x) - x + x = N(-x) + x = N(x) + x = x.$$

5. By part 1
$$-(x+y)+x+y = N(x+y) = N(x)+N(y) = -x+x-y+y = -x-y+x+y.$$
Then by Proposition 4.8
$$-(x + y) + N(x + y) = -x - y + N(x + y).$$
Again using part 1 one obtains
$$-(x+y)+N(-(x+y)) = -x-y+N(-x)+N(-y) = -x+N(-x)-y+N(-y).$$
Hence $-(x + y) = -x - y$.

6. By part 4 we have
$$N(x) = -x + x = -x - (-x) = -(x - x) = -N(x).$$ □

4.2 Flexible involutive meadows are varieties

In [1], Bergstra and Bethke studied the relations between involutive meadows and varieties. One of their results is that involutive meadows are varieties. We prove that flexible involutive meadows are also varieties.

Let us start by recalling the definition of varieties in this context, following [10].

Definition 4.10. If $\mathcal{F}$ is a signature, then an *algebra* A of type $\mathcal{F}$ is defined as an ordered pair (A, F), where A is a nonempty set and F is a family of finitary operations on A in the language of $\mathcal{F}$ such that, for each n-ary function symbol f in $\mathcal{F}$, there is an n-ary operation f^A on A.

Definition 4.11. A nonempty class K of algebras of the same signature is called a *variety* if it is closed under subalgebras, homomorphic images, and direct products.

A result by Birkhoff entails that K is a variety if and only if it can be axiomatized by identities.

Definition 4.12. Let Σ be a set of identities over the signature $\mathcal{F}$; and define $M(\Sigma)$ to be the class of algebras A satisfying Σ. A class K of algebras is an *equational class* if there is a set of identities Σ such that $K = M(\Sigma)$. In this case we say that K is defined, or axiomatized, by Σ.

Theorem 4.13. *K is a variety if and only if it is an equational class.*

The class K of flexible involutive meadows is axiomatized by the identities in Figure 3 over the signature $\Sigma = \{+, \cdot, -, ^{-1}, 0, 1, N(\cdot)\}$, where N is a unary function that, when interpreted with the external numbers, corresponds to the neutrix part of a number x. As a consequence of Birkhoff's theorem, we have the following result.

Corollary 4.14. *Flexible involutive meadows are varieties.*

4.3 Flexible involutive meadows and commutative von Neumann regular rings

In the investigation of meadows, the relation with commutative von Neumann regular rings with a multiplicative identity element seems to be of particular interest [3, 4].

We recall that a semigroup $(S, \cdot)$ is said to be *Von Neumann regular* if

$$\forall x \in S \, \exists y \in S \, (x \cdot x \cdot y = x) \, .$$

A commutative von Neumann regular ring with a multiplicative identity is a Von Neumann regular commutative semigroup for both addition and multiplication.

Flexible involutive meadows are also commutative von Neumann regular rings with a multiplicative identity element.

Proposition 4.15. *Let M be a flexible involutive meadow. Then M is a Von Neumann regular commutative semigroup for both addition and multiplication.*

Proof. This is a simple consequence of associativity together with axioms (FI_4) and (FI_{10}). □

In [3, Lemma 2.11] it was shown that commutative von Neumann regular rings can be expanded in a unique way to an involutive meadow. Since involutive meadows are also flexible involutive meadows, von Neumann regular rings can be expanded to flexible involutive meadows. The expansion to flexible involutive meadows might not be unique, though, due to the presence of different error terms.

Further research on the connection between commutative von Neumann regular rings and flexible involutive meadows goes beyond the scope of this paper.

4.4 Solids are flexible involutive meadows

We finish this section by showing that instead of working with the external numbers, one can use a purely algebraic approach by working with a structure called a *solid*. A solid is a generalization of the notion of field in which there are generalized neutral elements for both addition and multiplication (e and u, respectively), and generalized inverses (s and d). For a full list of the solid axioms and for the definitions of the functions e, u, s and d we refer to the appendix in [18] or [19].

The following proposition compiles some results from [19, Propositions 2.12, 4.8 and Theorem 2.16] and [16, Proposition 4.12], which we use to show that solids are flexible involutive meadows.

Proposition 4.16. *Let S be a solid and let $x, y, z \in S$.*

1. *If $x = e(x)$, then $e(xy) = e(x)y$;*
2. *If $x \neq e(x)$, then $u(x)e(x) = xe(u(x)) = e(x)$;*
3. *$x(z + e(y)) = xz + xe(y)$;*
4. *If $x \neq e(x)$, then $d(d(x)) = x$.*

Theorem 4.17. *Every solid is a flexible involutive meadow.*

Proof. Let S be a solid. Most of the axioms of flexible involutive meadows are also axioms of solids, by considering $N(x) = e(x)$, $-x = s(x)$, $x^{-1} = d(x)$, and $1 = u$. The only non-obvious cases are the cases of axioms (FI_7), (FI_9) and (FI_{10}).

For axiom (FI_7), if $x \neq e(x)$, using the solid axioms and Proposition 4.16 we obtain

$$(1 + N(x) \cdot x^{-1}) \cdot x = x + e(x)d(x)x = x + e(x)u(x) = x + e(x) = x.$$

If $x = e(x)$, the result follows from Lemma 4.5(1).

Axiom (FI_9) follows from Lemma 4.5(2), if $x = e(x)$ and from Proposition 4.16(4).

Finally, axiom (FI_{10}) follows easily from the solid axioms if $x \neq e(x)$ and from Lemma 4.5(3) if $x = e(x)$. □

As it turns out, and as mentioned above, one is not forced to work in a nonstandard setting. Indeed, any non-archimedean ordered field yields a model of flexible involutive meadows.

Let $\mathbb{F}$ be a non-archimedean ordered field. Let $\mathcal{C}$ be the set of all convex subgroups for addition of $\mathbb{F}$ and Q be the set of all cosets with respect to the elements of $\mathcal{C}$. In [18] the elements of $\mathcal{C}$ were called *magnitudes* and Q was called the *quotient class* of $\mathbb{F}$ with respect to $\mathcal{C}$. In the same paper it was also shown that the quotient class of a non-archimedean field is a solid. Hence we have following corollary.

Corollary 4.18. *The quotient class of a non-archimedean ordered field is a flexible involutive meadow.*

5 Further models for meadows using external numbers

In the introduction we claimed that the external numbers are particularly suitable for expressing the kind of concepts involved in the definition of the inverse of zero. In order to support that claim, we explore further models for meadows inspired by the external numbers. We start by building a model for flexible involutive meadows over a finite field $\mathbb{F}$ and proceed by constructing a model for common meadows over ${}^*\mathbb{R}$.

5.1 Finite models of flexible involutive meadows

In this subsection we show that any finite field can be extended to a finite model of a flexible involutive meadow.

Recall that a finite field is isomorphic to $\mathbb{F}_{p^m}$, with p a prime number and m a positive integer. As a consequence, without loss of generality we assume that elements of the finite field, which we will simply denote $\mathbb{F}$ from here on, are of the form $a \mod p^m$ with $a \in \mathbb{N}$, so we can identify elements of $\mathbb{F}$ with natural numbers between 0 and $p^m - 1$.

Definition 5.1. Let $(\mathbb{F}, +, \cdot)$ be a finite field. Without loss of generality, we may think that the elements of $\mathbb{F}$ are natural numbers between 0 and $|\mathbb{F}| - 1$. We define $(\widehat{\mathbb{F}}, \oplus, \odot)$ as follows.

- For every $a \in \mathbb{F}$, we define the external number $\widehat{a} = a + \oslash$ and $\widehat{\mathbb{F}} = \{\widehat{a} : a \in \mathbb{F}\}$.
- For every nonzero $a \in \mathbb{F}$, we set $(\widehat{a})^{-1} = \widehat{a^{-1}}$.
- $(\widehat{0})^{-1} = \widehat{0}$ (this definition is motivated by, and indeed coincides with the one in Definition 4.2).
- The sum $\oplus$ and product $\odot$ over $\widehat{\mathbb{F}}$ are defined as:

$$\widehat{a} \oplus \widehat{b} = \widehat{a + b}$$

and

$$\widehat{a} \odot \widehat{b} = \widehat{ab}$$

where sums and product on the right hand side are the sums and product in $\mathbb{F}$.

Theorem 5.2. *Let $\mathbb{F}$ be a finite field. Then $\widehat{\mathbb{F}}$ satisfies the axioms for flexible involutive meadows.*

Proof. If $x \neq \widehat{0}$, the axioms are satisfied since $\mathbb{F}$ is a field and the operations in $\widehat{\mathbb{F}}$ are compatible with the ones in $\mathbb{F}$.

If $x = \widehat{0}$, axioms $(\mathrm{I}_1) - (\mathrm{I}_8)$ follow from the fact that $\mathbb{F}$ is a field.

As for axiom (I_9), if $x = \widehat{0}$, then $\widehat{0}^{\ -1} = \widehat{0}$, so that $(\widehat{0}^{\ -1})^{-1} = \widehat{0}^{\ -1} = \widehat{0}$.

Finally, for axiom (I_{10}), if $x = \widehat{0}$, then $\widehat{0} \cdot \widehat{0}^{\ -1} = 0$. □

Remark 5.3. Similar models for involutive meadows can be obtained without recurring to infinitesimals, as one could define the alternative model $\tilde{\mathbb{F}}$ by requiring the existence of an element $E \notin \mathbb{F}$ and defining, for each $a \in \mathbb{F}$ the element $\tilde{a} = a + E$ and the set $\tilde{\mathbb{F}} := \{\tilde{a} : a \in \mathbb{F}\}$ with the operations

$$\tilde{a} \oplus \tilde{b} = (a + E) + (b + E) = (a + b) + E,$$

(note that, in particular $E \oplus E = (0 + E) + (0 + E) = (0 + 0) + E = 0 + E = E$),

$$\tilde{a} \odot \tilde{b} = (a + E) \cdot (b + E) = (a \cdot b) + E$$

and

$$\tilde{a}^{-1} = a^{-1} + E$$

and, finally,

$$\tilde{0}^{-1} = 0 + E = \tilde{0}.$$

5.2 A model for common meadows based on $\mathbb{R}$

In this section we introduce a model $\widehat{\mathbb{R}}$ for the axioms of common meadows given in Figure 2. In our model, we consider the elements of $\mathbb{R}$ plus an error term in the form of a neutrix. In order to make things concrete, we choose to use the neutrix $\oslash$ but, in principle, any neutrix included in $\oslash$ can be used.

Elements of $\mathbb{R}$ will be represented by external numbers of the form $r + \oslash$ with $r \in \mathbb{R}$, while ${}^*\mathbb{R}$ will act as an inverse of the neutrix $\oslash$, which is the representative of 0. We can interpret $\oslash^{-1} = {}^*\mathbb{R}$ as the smallest neutrix collecting all the inverses of the elements of $\oslash$. The fact that the inverse of 0 has, in a sense, the maximum possible uncertainty is in good accord with both the intuition that division by 0 introduces an error term, and to the common practice of having the inverse of 0 not being a member of the original field [5].

The inverse of ${}^*\mathbb{R}$ is again ${}^*\mathbb{R}$. This choice can be justified in two ways. We can again interpret ${}^*\mathbb{R}^{-1}$ as the smallest neutrix collecting all the inverses of the elements of ${}^*\mathbb{R}$ or, alternatively, since $\oslash \subset {}^*\mathbb{R}$, the inverse of ${}^*\mathbb{R}$ should also be maximal.

Definition 5.4. We define the set $\widehat{\mathbb{R}}$ as follows:

- For every $r \in \mathbb{R}$, we set $\widehat{r} = r + \oslash \in \widehat{\mathbb{R}}$.
- ${}^*\mathbb{R} \in \widehat{\mathbb{R}}$.
- For every nonzero $r \in \mathbb{R}$, we set $\widehat{r}^{-1}$, with the quotient introduced in Definition 3.4 (see also Proposition 3.5).
- We define $(\widehat{0})^{-1} = {}^*\mathbb{R}$ and ${}^*\mathbb{R}^{-1} = {}^*\mathbb{R}$ (so ${}^*\mathbb{R}$ acts like the error term $\mathbf{a}$ in the definition of common meadow).
- The sum and product over $\widehat{\mathbb{R}}$ are the Minkowski operations introduced in Definition 3.3.

In the definition of real numbers as limits of Cauchy sequences, real numbers can be seen as being determined up to "infinitesimals", which have an interpretation in terms of sequences converging to 0. In the model of common meadows introduced in the previous definition, this idea is expressed by the representation of r as $\hat{r} = r + \oslash$.

An immediate consequence of the previous definition is that for every $x \in \widehat{\mathbb{R}}$ we have

$$x + (\widehat{0})^{-1} = (\widehat{0})^{-1} + x = x \cdot (\widehat{0})^{-1} = (\widehat{0})^{-1} \cdot x = {}^*\mathbb{R}. \tag{5.1}$$

In the next lemma, we establish that the operations in $\widehat{\mathbb{R}}$ are compatible with those in $\mathbb{R}$.

Lemma 5.5. *For every $r, s \in \mathbb{R}$,*

- $\widehat{r + s} = \widehat{r} + \widehat{s}$*;*
- $\widehat{r \cdot s} = \widehat{r} \cdot \widehat{s}$*.*

Moreover, for every nonzero $r \in \mathbb{R}$, $\widehat{r}^{-1} = \widehat{r^{-1}}$.

Proof. The first two properties are a consequence of the following equalities

$$\widehat{r+s} = r + s + \oslash = (r + \oslash) + (s + \oslash) = \widehat{r} + \widehat{s}.$$

and, taking into account that r and s are real numbers,

$$\widehat{r \cdot s} = r \cdot s + \oslash = (r + \oslash) \cdot (s + \oslash) = \widehat{r} \cdot \widehat{s}.$$

As for the inverse, if $r \neq 0$, $\widehat{r^{-1}} = \frac{1}{r} + \oslash$, whereas, by Proposition 3.5,

$$\widehat{r}^{-1} = (r + \oslash)^{-1} = r^{-1} + r^{-2} \cdot \oslash = \frac{1}{r} + \oslash. \qquad \square$$

Corollary 5.6. *For every $r \in \mathbb{R}$,*

1. *if $r \neq 0$, then $\widehat{r} \cdot \widehat{r}^{-1} = \widehat{1}$;*
2. *if $r \neq 0$, then $\widehat{0} \cdot \widehat{r}^{-1} = \widehat{0}$;*
3. *$\widehat{1} = \widehat{1} + \widehat{0} \cdot \widehat{r}$;*
4. *$\widehat{r} + {}^*\mathbb{R} = {}^*\mathbb{R} + \widehat{r} = {}^*\mathbb{R}$ and $\widehat{r} \cdot {}^*\mathbb{R} = {}^*\mathbb{R} \cdot \widehat{r} = {}^*\mathbb{R}$;*
5. *${}^*\mathbb{R} + {}^*\mathbb{R} = {}^*\mathbb{R} - {}^*\mathbb{R} = {}^*\mathbb{R} \cdot {}^*\mathbb{R} = {}^*\mathbb{R}$.*

Equalities (1)–(3) in Corollary 5.6 can be obtained from the corresponding equalities for real numbers by repeated use of Lemma 5.5.

Theorem 5.7. *$\widehat{\mathbb{R}}$ is a model of axioms (M_1)–(M_{14}) of common meadows.*

Proof. We start by showing that axiom (M_1) is satisfied. If x, y and z are different from ${}^*\mathbb{R}$, then this is a consequence of Lemma 5.5. If at least one of x, y and z is equal to ${}^*\mathbb{R}$, then both sides of the equality evaluate to ${}^*\mathbb{R}$ by Corollary 5.6. The proof follows similar steps for axioms (M_2) – (M_9). Note that, for axiom (M_8), we use also the fact that we have only one order of magnitude besides ${}^*\mathbb{R}$.

Let us show that axiom (M_{10}) is satisfied. If $x = \widehat{0}$:

$$\widehat{0} \cdot (\widehat{0})^{-1} = {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R}.$$

If $x = {}^*\mathbb{R}$: as a consequence of the definition of ${}^*\mathbb{R}^{-1}$, we have

$${}^*\mathbb{R} \cdot {}^*\mathbb{R}^{-1} = {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R}^{-1}.$$

We now turn to axiom (M_{11}). If x, y are not equal to 0 nor to ${}^*\mathbb{R}$, the axiom holds as a consequence of Lemma 5.5. Otherwise, due to the definition of the inverse, both sides are equal to ${}^*\mathbb{R}$.

For axiom (M_{12}), if $x = \widehat{r}$ for some $r \in \mathbb{R}$, the axiom holds as a consequence of Corollary 5.6 (3). If $x = {}^*\mathbb{R}$, then

$$(\widehat{1} + \widehat{0} \cdot {}^*\mathbb{R})^{-1} = (1 + {}^*\mathbb{R})^{-1} = {}^*\mathbb{R}^{-1} = {}^*\mathbb{R} = \widehat{1} + {}^*\mathbb{R} = \widehat{1} + \widehat{0} \cdot {}^*\mathbb{R}.$$

Axiom (M_{13}) is satisfied as a consequence of the definition of $(\widehat{0})^{-1}$. Finally, axiom (M_{14}) is satisfied as a consequence of (5.1). $\square$

6 Final remarks and open questions

In this paper we have introduced the notion of flexible involutive meadow, by means of an equational axiomatization, and constructed some models based on the external numbers and non-archimedean fields. We have also shown a model for common meadows based on the real numbers and of involutive meadows based on finite fields. We would like to point out that, with similar techniques, one could also obtain meadows based on rational numbers.

The model for common meadows developed in Section 5.2 suggests that it is possible to study a flexible version of common meadows, in the spirit of what has been done in Section 4 for involutive meadows. In order to do so, it is possible to adapt the axioms by replacing 0 with $N(x)$, where $N(x)$ is an error term analogous to that of flexible involutive meadows.

In the context of external numbers, where $N(x)$ is the neutrix part of x, and for zeroless x, one has the inclusion $N(x) \subseteq x \cdot \oslash$. This grounds the interpretation of the flexible counterparts of axioms (M_4), (M_{10}) and (M_{12}). Moreover, as in the model discussed in Section 5.2, the element **a** can be taken as ${}^*\mathbb{R}$.

Section 4.4 unveils a connection between the two apparently very different algebraic structures of solids and flexible involutive meadows. This connection is in line with other works connecting algebraic structures related with meadows and structures arising in the context of nonstandard analysis (see [11, 15]). We believe that this line of research is worth exploring in future work.

To conclude, we mention some other possible directions of future work.

We would like to study related variants of meadows as well as their algebraic properties. For example, the study of *flexible* cancellation meadows, i.e. meadows in which the multiplicative cancellation axiom

$$x \neq 0 \wedge x \cdot y = x \cdot z \Rightarrow y = z$$

or its flexible counterpart (where we substitute 0 by an error term e) holds; or *flexible* arithmetical meadows in the sense of [4]; or *flexible* meadows of rational numbers (see e.g. [6]). Are flexible arithmetical meadows, i.e. flexible meadows satisfying

(A_1) and (A_2) (necessarily) connected with nonstandard models of arithmetic? As for the flexible meadows of rational numbers, are they a minimal algebra? If so, that might provide a connection with data types.

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Kanger-Wang-type Sequent Calculi with Equality

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Abstract

We develop a proof theoretic analysis of several extensions of the **G3**[**mic**] sequent calculi with rules for equality related to those introduced by H.Wang and S.Kanger. In the classical case we relate our results with the semantic tableau method for first order logic with equality. In particular we establish that, for languages without function symbols, in Fitting's alternate semantic tableau method, strictness (which does not allow to retain the formulae that are modified) can be imposed together with the orientation of the replacement of equals provided the latter is allowed on all atomic formulae and their negations. Furthermore we prove that the result holds also for languages with function symbols provided strictness is not imposed on equalities, leaving it open whether or not strictness can be imposed on equalities as well. Finally we discuss to what extent the strengthened form of the nonlengthening property of Orevkov known to hold for the sequent calculi with the structural rules applies also to the present context.

Keywords: Sequent Calculus, Structural Rules, Equality, Replacement Rules, Admissibility

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1 Introduction

In [12] we have shown that full cut elimination holds for the extension of Gentzen's sequent calculi obtained by adding the Reflexivity Axiom $\Rightarrow t = t$, and the left introduction rules for =:

$$\frac{\Gamma \Rightarrow \Delta, F[x/r]}{r = s,\ \Gamma \Rightarrow \Delta, F[x/s]} =_1 \qquad \frac{\Gamma \Rightarrow \Delta, F[x/r]}{s = r,\ \Gamma \Rightarrow \Delta, F[x/s]} =_2$$

where F is a formula; $F[x/r]$ and $F[x/s]$, as in [16], denote the result of the replacement in F of all free occurrences of x by r or s and Γ, Δ are finite multisets of formulae, with $|\Delta| = 0$ in the intuitionistic case.

These rules correspond directly to the natural deduction rules for equality: in particular $=_1$ and $=_2$ are the left introduction rules corresponding to the two, equally natural, elimination rules for =. For that reason we consider the result of adding them to LK and LJ to be the basic sequent calculi for first order logic with equality, against which any other should be compared. Thus, the adequacy of calculi free of structural rules requires that they be admissible in the system.

In [12] the above cut-elimination result is extended to other well motivated calculi with rules where $F[x/r]$ and $F[x/s]$ occur in the antecedent of the premiss and of the conclusion. The purpose of this work is to establish the adequacy of corresponding systems free of structural rules, some of which, in the classical case, are of particular interest in connection with the semantic tableau method for first order logic with equality. For that we have to refer to systems of that sort as far as logic is concerned such as the multisuccedent systems for minimal, intuitionistic and classical logic originated with Dragalin's [7] and denoted by m-**G3**[**mic**] in [16], that we will adopt as our logical systems. Since we will be dealing exclusively with such multisuccedent systems, as remarked in [16] (pg. 83), the prefix m- is redundant and we will drop it. Thus **G3i** will denote the multisuccedent **G3** calculus for intuitionistic logic, **G3m** the analogous calculus for minimal logic, **G3c** the classical calculus and **G3**[**mic**] any of such three calculi. We then adopt the Reflexivity Axiom in the form $\Gamma \Rightarrow t = t$, to be denoted by $\overline{\text{Ref}}$; restrict the formula F in $=_1$ and $=_2$ to be atomic and note that it sufficient, as well as necessary, to repeat the principal formula $r = s$ in the antecedent of the premiss of the rule to obtain what may be considered a most natural sequent calculus with equality free of structural rules. Actually such rules are particular cases of those introduced in the classical case and shown, taken together with the reflexivity axiom, to be semantically complete by H.Wang in [18]. We will denote with Rep_1^r and Rep_2^r the rules so obtained, namely:

$$\frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Rep}_1^r \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Rep}_2^r$$

where P is atomic (possibly an equality), called the *context* formula, while $r = s$ ($s = r$) is called the *operating equality*, r (s) the *input* (*output*) term and $P[x/r]$ ($P[x/s]$) the *input* (*output*) formula. In the classical case, the well known connection between such kind of calculi and the semantic tableau method for first order logic with equality developed for example in [9] and [8], add motivations to those in [12], for the rules:

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep}_1^l \qquad \frac{s = r, P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep}_2^l$$

Rep_2^l corresponds to branch expansions in which *strictness* is required, namely the formula $P[x/s]$ in which the term replacement is operated in is checked out. When strictness, as in [8], is not required, adopting for Rep the notation in [16], the corresponding rules are:

$$\frac{r = s, P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep'} \qquad \frac{s = r,\ P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{s = r,\ P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep}$$

Rep corresponds to the Tableau Replacement Rule used in [8] pg. 289 together with the Tableau Reflexivity Rule that allows the expansion of a branch by the addition of any identity $t = t$.

Our results will be based on the following fact that follows from the main result in [13]: for any set $\mathcal{R}$ of atomic rules for equality that we will consider, if the structural rules are admissible in $\mathcal{R}$, identified with the calculus that consists of the initial sequents, including $\bot, \Gamma \Rightarrow \Delta$ in the intuitionistic and classical case, and the rules in $\mathcal{R}$, then they are admissible also in the calculus $\mathbf{G3[mic]}^{\mathcal{R}}$ obtained by adding the rules in $\mathcal{R}$ to $\mathbf{G3[mic]}$.

2 Preliminaries on the logical calculi and Equality Rules and Systems

The sequent calculus denoted by **G3c** in [16] (pg 83), has the following initial sequents and rules, where P is an atomic formula and A, B stand for any formula in a first order language (function symbols included) with bound variables distinct from the free ones, and Γ and Δ are finite multisets of formulae:

Initial sequents

$$P, \Gamma \Rightarrow \Delta, P$$

Logical rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ R\to$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{A[x/t], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{A[x/a], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

In **G3i** the rules $L\to$, $R\to$ and $R\forall$ are replaced by:

$$\frac{A \to B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L^i\to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}\ R^i\to$$

$$\frac{\Gamma \Rightarrow A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A}\ R^i\forall$$

Finally **G3m** is obtained from **G3i** by replacing $L\bot$ by the initial sequents $\bot, \Gamma \Rightarrow \Delta, \bot$.

In all such systems a is a free variable that does not occur in the conclusion of $L\exists$ and $R\forall$.

G3[**mic**] denotes any of the systems **G3m**, **G3i** or **G3c**.

The left and right weakening rules, LW and RW have the form:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathrm{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathrm{RW}$$

The left and right contraction rules, LC and RC have the form:

$$\frac{A.A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathrm{LC} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathrm{RC}$$

$\mathrm{LC}^{=}$ is the rule LC in which the contracted formula A is an equality.
The cut rule has the form:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}\ \mathrm{Cut}$$

Weakening, contraction and cut are the structural rules whose admissibility we are going to investigate.

In consequence of the more general result concerning the addition of atomic rules to the above sequent calculi established in [13], for any set $\mathcal{R}$ of the equality rules in the Introduction and the further single premiss equality rules to be introduced in the sequel we have the following:

Theorem 1. *[Theorem 1 in [13]] If the structural rules are admissible in $\mathcal{R}$, then they are admissible in* $\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ *as well.*

that will be instrumental for the present work.

A further rule that will play an important auxiliary role is the following *congruence rule*:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, r = s \qquad \Gamma_2 \Rightarrow \Delta_2, P[x/r]}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, P[x/s]}\ \mathrm{CNG}$$

Note The rule *CNG* is among those used in the extension of the system CERES in [1], pg.170.

2.1 Equality rules and systems

For easier reading of the paper we collect the rules in the Introduction as well as those to be introduced in the sequel:

$$\frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Rep}_1^r \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{Rep}_2^r$$

$$\frac{r = s, P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep}_1^l \qquad \frac{s = r, P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep}_2^l$$

$$\frac{r = s, P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{r = s, P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep'} \qquad \frac{s = r,\ P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{s = r,\ P[x/s], \Gamma \Rightarrow \Delta}\ \text{Rep}$$

- $\text{Rep}_1^{r=r}$ and $\text{Rep}_2^{r=r}$ are Rep_1^r and Rep_2^r subject to the restriction that when the context formula is an equality only its right-handside is modified.
- $\text{Rep}_1^{r/=}$ and $\text{Rep}_2^{r/=}$ are Rep_1^r and Rep_2^r restricted to context formulae that are equalities.
- $\text{Rep}_1^{l/(=)}$ and $\text{Rep}_2^{l/(=)}$ are Rep_1^l and Rep_2^l restricted to context formulae that are not equalities.
- Rep_1^{l+} and Rep_2^{l+} are Rep_1^l and Rep_2^l extended with the rules:

$$\frac{s = r, E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{s = r, E[x/s], \Gamma \Rightarrow \Delta} \quad \text{and} \quad \frac{r = s,\ E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{r = s,\ E[x/s], \Gamma \Rightarrow \Delta}$$

where E is an equality.

The notation of the various systems to be considered are going to reflect directly the rules that they contain. For instance:

$$\begin{aligned}
\mathcal{R}_{12}^{r} &= \{\overline{\text{Ref}}, \text{Rep}_1^r, \text{Rep}_2^r\} \\
\mathcal{R}_{12}^{rl} &= \{\overline{\text{Ref}}, \text{Rep}_1^r, \text{Rep}_2^r, \text{Rep}_1^l, \text{Rep}_2^l\} \\
\mathcal{R}_{2}^{rl} &= \{\overline{\text{Ref}}, \text{Rep}_2^r, \text{Rep}_2^l\} \\
\mathcal{R}_{12}^{r=r} &= \{\overline{\text{Ref}}, \text{Rep}_1^{r=r} \text{Rep}_2^{r=r}\} \\
\mathcal{R}_{12}^{r/=,l/(=)} &= \{\overline{\text{Ref}},\ \text{Rep}_1^{r/=}, \text{Rep}_2^{r/=}, \text{Rep}_1^{l/(=)}, \text{Rep}_2^{l/(=)}\} \\
\mathcal{R}_{12}^{rl} &= \{\overline{\text{Ref}}, \text{Rep}_1^r, \text{Rep}_2^r, \text{Rep}_1^l, \text{Rep}_2^l\} \\
\mathcal{R}_{2}^{rl+} &= \{\overline{\text{Ref}}, \text{Rep}_2^r, \text{Rep}_2^{l+}\}
\end{aligned}$$

3 Basic results concerning the equality rules

3.1 Admissibility of Weakening and Right Contraction

The weakening rules are clearly height preserving admissible in the systems consisting of $\overline{\text{Ref}}$ and some of the equality rules. The single premiss equality rules modify at most one formula in the succedent of their premiss. Furthermore the initial sequents and those in $\overline{\text{Ref}}$ remain initial sequents or in $\overline{\text{Ref}}$ if all the formulae in their succedent, except the principal one, are eliminated. By a straightforward induction on the height of derivations it follows that if $\Gamma \Rightarrow \Delta$ has a derivation in the systems we are considering, then there is a formula A in Δ such that $\Gamma \Rightarrow A$ has a derivation of the same height. That is the case also for the two premisses rule CNG that eliminates a formula from the succedent of its first premiss and modifies a single formula of the succedent of the second. As a consequence the right contraction rule is height-preserving admissible in all the systems we are going to deal with.

3.2 Basic equivalence property

A basic tool for our investigation is provided by the following proposition, where by an equality rule we mean any of the rules presented in the introduction other than $\overline{\text{Ref}}$:

Proposition 2. *All the equality rules are equivalent in* $\{\overline{\text{Ref}}$, Cut, LC$\}$.

Proof We first show that if we add any one of the equality rules to $\{\overline{\text{Ref}}$, Cut, LC$\}$, then the following rule of Left Symmetry becomes derivable:

$$\frac{r = s, \Gamma \Rightarrow \Delta}{s = r, \Gamma \Rightarrow \Delta}\ \text{Symm}$$

Case 1.1. The rule added is Rep^r_1. Then we have the following derivation of Symm:

$$\cfrac{\cfrac{s = r \Rightarrow s = s}{s = r \Rightarrow r = s} \qquad r = s, \Gamma \Rightarrow \Delta}{s = r, \Gamma \Rightarrow \Delta}\ \text{Cut}$$

Case 1.2. The rule added is Rep^r_2. Similar to Case 1.1

Case 2.1. The rule added is Rep^l_1. Then we have the following derivation:

$$\dfrac{\Rightarrow r=r \qquad \dfrac{\dfrac{\dfrac{r=s,\Gamma\Rightarrow\Delta}{r=s,\ r=r,\Gamma\Rightarrow\Delta}\,\text{LW}}{r=s,\ s=r,\Gamma\Rightarrow\Delta}\,\text{Rep}_1^l}{r=r,\ s=r,\Gamma\Rightarrow\Delta}\,\text{Rep}_1^l}{s=r,\Gamma\Rightarrow\Delta}\,\text{Cut}$$

Case 2.2. The rule added is Rep_2^l. Then the derivation is the same as for case 2.1, except that LW introduces $s=s$ and Rep_2^l is used instead of Rep_1^l.

Case 3.1. The rule added is Rep. Then we have the following derivation:

$$\dfrac{\Rightarrow s=s \qquad \dfrac{\dfrac{r=s,\Gamma\Rightarrow\Delta}{r=s,s=s,s=r,\Gamma\Rightarrow\Delta}\,\text{LW}}{s=s,s=r,\Gamma\Rightarrow\Delta}\,\text{Rep}}{s=r,\Gamma\Rightarrow\Delta}\,\text{Cut}$$

Case 3.2 The rule added is Rep'. Similar to Case 3..

Case 4. The rule added is CNG. Then we have the following derivation:

$$\dfrac{\dfrac{s=r\Rightarrow s=r \qquad \Rightarrow s=s}{s=r\Rightarrow r=s} \qquad r=s,\Gamma\Rightarrow\Delta}{s=r,\Gamma\Rightarrow\Delta}\,\text{Cut}$$

Clearly the derivability of Symm makes equivalent the rules of the same type with index 1 and 2. Thus it suffices to verify the equivalence (that does not depend on the availability of Symm) between Rep_1^r and Rep_2^l; Rep_1^l and Rep; Rep_1^r and CNG. We leave the easy details to the reader. □.

Corollary 3. *All the systems* $\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$*, for* $\mathcal{R}$ *that consists of* $\overline{\text{Ref}}$ *and of some of the equality rules and such that the structural rules are admissible in* $\mathcal{R}$*, are equivalent.*

4 Admissibility of the structural rules

4.1 Necessity of the repetition of the operating equalities in the premiss of the equality rules

We show that, as stated in the introduction, the addition of $\overline{\text{Ref}}$, $=_1$ and $=_2$ to **G3**[**mic**] is not sufficient to yield appropriate extensions free of structural rules.

Actually even if, beside $\overline{\text{Ref}}$, $=_1$ and $=_2$, also the Cut rule is added, in the resulting system the left contraction rule remains not admissible.

Let $\mathcal{R} = \{\overline{Ref}, =_1, =_2, \text{Cut}\}$. We will prove that LC is not admisssible in $\mathbf{G3c}^{\mathcal{R}}$ by showing that the following sequent:

$$*)\ \ a = f(a) \Rightarrow a = f(f(a))$$

whose expansion $a = f(a), a = f(a) \Rightarrow a = f(f(a))$ is immediately derivable by means of an $=_2$-inference applied to $a = f(a) \Rightarrow a = f(a)$, is not derivable in $\mathcal{R}$. In fact if $*)$ were derivable in $\mathbf{G3c}^{\mathcal{R}}$ (as for Proposition 7 in [13]) $*)$ would have a derivation in which no logical inference different from $L\bot$ precedes a $=_1$, $=_2$ or Cut-inference. As a consequence $*)$ would be derivable in $\mathcal{R}$ itself, which is impossible.

In order to show that $*)$ is not derivable in $\mathcal{R}$, we first note that if a sequent $\Gamma \Rightarrow r = s$ is derivable in $\mathcal{R}$, then the sequent $\Gamma_= \Rightarrow r = s$, where $\Gamma_=$ denotes the multiset of equalities in Γ, has a derivation in $\mathcal{R}$ that involves only equalities. An easy induction on the height of such derivations shows that if Γ is a multiset of *identities* i..e equalities of the form $r = r$ and $\Gamma \Rightarrow r = s$ is derivable in $\mathcal{R}$ then $r = s$ is itself an identity ($r \equiv s$). That being noted, we prove the following:

Proposition 4. *If Γ is a multiset of identities and $E, \Gamma \Rightarrow E'$ is derivable in $\mathcal{R}$, where E' coincides with $a = f(f(a))$ or with $f(f(a)) = a$, then also E has one of such two forms. Hence $a = f(a) \Rightarrow a = f(f(a))$ is not derivable in $\mathcal{R}$.*

Proof We proceed by induction on the height of a derivation $\mathcal{D}$ in $\mathcal{R}$ of $E, \Gamma \Rightarrow E'$.

If $h(\mathcal{D}) = 0$, then $E, \Gamma \Rightarrow E'$ must be an initial sequent and E coincides with E' so that the claim is trivial.

If $h(\mathcal{D}) > 0$ and $\mathcal{D}$ ends with an $=_1$ inference that introduces E in the antecedent, then $\mathcal{D}$ has the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma \Rightarrow r = s\end{array}}{E, \Gamma \Rightarrow E'}$$

By the previous remark $r = s$ is an identity $r = r$ and we note that the only possibilities of obtaining E' by a substitution applied to $r = r$ is that r coincides with a or with $f(f(a))$ in which cases E is necessarily $a = f(f(a))$ or $f(f(a)) = a$.

The same argument applies if $\mathcal{D}$ ends with an $=_2$-inference introducing E.

If $\mathcal{D}$ ends with an $=_1$ or $=_2$-inference introducing a formula in Γ, which is therefore an identity, the conclusion is a trivial consequence of the induction hypothesis.

If $\mathcal{D}$ ends with a Cut, we have two cases.

Case1. $\mathcal{D}$ has the form:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ A, E, \Gamma_2 \Rightarrow E'\end{matrix}}{E, \Gamma_1, \Gamma_2 \Rightarrow E'}$$

In this case, looking at $\mathcal{D}_0$ we have that A is itself an identity so that it suffices to apply the induction hypothesis to $\mathcal{D}_1$ to conclude that E is $a = f(f(a))$ or $f(f(a)) = a$.

Case 2 $\mathcal{D}$ has the form:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ E, \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ A, \Gamma_2 \Rightarrow E'\end{matrix}}{E, \Gamma_1, \Gamma_2 \Rightarrow E'}$$

By the induction hypothesis applied to $\mathcal{D}_1$ A has one of the two forms $a = f(f(a))$ or $f(f(a)) = a$ so that it suffices to apply the induction hypothesis to $\mathcal{D}_0$ to conclude that the same holds for E.

That $a = f(a) \Rightarrow a = f(f(a))$ is not derivable in $\mathcal{R}$ follows by letting Γ be the empty set and E' the equality $a = f(f(a))$. $\square$

4.2 Sufficiency of the repetition of the operating equalities in the premiss

We now prove that the repetition of the operating equalities in the premiss of the $=_1$ and $=_2$-rules, which yields the Rep_1^r and Rep_2^r rules, suffices to yield a system, indeed a very natural one, for which the structural rules are admissible.

Theorem 5. *For $\mathcal{R}_{12}^r = \{\overline{\text{Ref}}, \text{Rep}_1^r, \text{Rep}_2^r\}$, the structural rules are admissible in* $\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}_{12}^r}$.

Proof By Theorem 1 it suffices to show that the structural rules are admissible in $\mathcal{R}_{12}^r$. The admissibility of LC is straightforward, since the rules of $\mathcal{R}_{12}^r$ do not change the antecedent of their premiss. For the admissibility of Cut we transform a given derivation $\mathcal{D}$ in $\mathcal{R}_{12}^r$ + Cut into a derivation $\mathcal{D}'$ in $\{\overline{\text{Ref}}, \text{LC}, \text{CNG}, \text{Cut}\}$ by using the following derivation of Rep_1^r from CNG and $\text{LC}^=$:

$$\dfrac{\dfrac{r = s \Rightarrow r = s \qquad r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{CNG}}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}\ \text{LC}^=$$

and the derivation of Rep_2^r from CNG and $\text{LC}^=$ that can be obtained from that of Rep_1^r thanks to the derivation of Symm from CNG shown in Case 4. of the proof of Proposition 2.

From $\mathcal{D}'$ we eliminate the applications of the Cut rule in order to obtain a derivation $\mathcal{D}''$ in $\{\overline{\text{Ref}}, \text{LC}, \text{CNG}\}$. To show that this is possible, because of the presence of the rule LC, we have to show that the following more general rule:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A^n, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}$$

where A^n denotes the multiset that contains A n times and nothing else, is admissible in $\{\overline{\text{Ref}}, \text{LC}, \text{CNG}\}$. That is shown by a straightforward induction on the height of the derivation of $A^n, \Lambda \Rightarrow \Theta$.

Then to obtain, from $\mathcal{D}''$, the desired cut-free derivation in $\mathcal{R}_{12}^r$ of the endsequent of $\mathcal{D}$, it suffices to exploit the admissibility of LC and CNG in $\mathcal{R}_{12}^r$. The admissibility of CNG in $\mathcal{R}_{12}^r + \text{LC}$, hence in $\mathcal{R}_{12}^r$, can be proved by induction on the height of the derivation of its first premiss (see [12] for the analogous result for the sequent calculi with structural rules). In fact let $\mathcal{D}$ be of the form:

$$\frac{\begin{matrix}\mathcal{D}_0 \\ \Gamma' \Rightarrow \Delta, r = s\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Lambda \Rightarrow \Theta, P[x/r]\end{matrix}}{\Gamma, \Lambda \Rightarrow \Delta, \Theta, P[x/s]}\ \text{CNG}$$

where $\mathcal{D}_0$ and $\mathcal{D}_1$ are derivations in $\mathcal{R}_{12}^r + \text{LC}$. We have to show that also the conclusion of $\mathcal{D}$ is derivable in $\mathcal{R}_{12}^r + \text{LC}$. If r and s coincide, then the conclusion is obtained by weakening the conclusion of $\mathcal{D}_1$. Assuming r is distinct from s, we proceed by induction on the height $h(\mathcal{D}_0)$ of $\mathcal{D}_0$.

If $h(\mathcal{D}_0) = 0$ and $\mathcal{D}_0$ is an initial sequent with principal formula common to Γ and Δ, then the conclusion of $\mathcal{D}$ is also an initial sequent and the given of CNG-inference can be eliminated, while if it is of the form $r = s, \Gamma' \Rightarrow \Delta, r = s$, then $\mathcal{D}$, namely

$$\frac{r = s, \Gamma' \Rightarrow \Delta, r = s \qquad \begin{matrix}\mathcal{D}_1 \\ \Lambda \Rightarrow \Theta, P[x/r]\end{matrix}}{r = s, \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/s]}$$

is transformed into:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1 \\ \Lambda \Rightarrow \Theta, P[x/r]\end{matrix}}{r = s,\ \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/r]}\ \text{LW}}{r = s, \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/s]}\ \text{Rep}_1^r$$

If $h(\mathcal{D}_0) > 0$ and $\mathcal{D}_0$ ends with an Rep^r_1- inference and the principal formula occurs in Δ then the derivation of the conclusion is obtained as a straightforward consequence of the induction hypothesis. On the other hand if the principal formula is $r = s$ of the form $r^\circ[x/q] = s^\circ[x/q]$, with $\mathcal{D}$ of the form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00}\\ p = q, \Gamma' \Rightarrow \Delta, r^\circ[x/p] = s^\circ[x/p]\end{matrix}}{p = q, \Gamma' \Rightarrow \Delta, , r^\circ[x/q] = s^\circ[x/q]} \qquad \begin{matrix}\mathcal{D}_1\\ \Lambda \Rightarrow \Theta, P[x/r^\circ[x/q]]\end{matrix}}{p = q, \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/s^\circ[x/q]]}$$

$\mathcal{D}$ can be transformed into:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00}\\ p = q, \Gamma' \Rightarrow \Delta, r^\circ[x/p] = s^\circ[x/p]\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Lambda \Rightarrow \Theta, P[x/r^\circ[x/q]]\end{matrix}}{p = q, \Lambda \Rightarrow \Theta, P[x/r^\circ[x/q]]}\,\text{LW}}{p = q, \Lambda \Rightarrow \Theta, P[x/r^\circ[x/p]]}\,\text{Rep}^r_2}{p = q,\ p = q,\ \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/s^\circ[x/p]]}\,\text{ind}}{p = q,\ p = q,\ \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/s^\circ[x/q]]}\,\text{Rep}^r_1}{p = q,\ \Gamma', \Lambda \Rightarrow \Delta, \Theta, P[x/s^\circ[x/q]]}\,\text{LC}^=$$

where ind means that, by induction hypothesis, the given derivations in $\mathcal{R}^r_{12} + \text{LC}$ of the sequents above the line can be transformed into a derivation in $\mathcal{R}^r_{12} + \text{LC}$ of the sequent below the line. If the premiss is obtained by an Rep^r_2 the argument is the same except that in the transformed derivation we use Rep^r_1 in place of Rep^r_2 and conversely. The case in which the first premiss is obtained by means of an LC-inference is straightforward. □

Corollary 6. $\mathcal{R}^{rl}_{12} = \{\overline{\text{Ref}}, \text{Rep}^l_1, \text{Rep}^l_2, \text{Rep}^r_1, \text{Rep}^r_2\}$ *and* $\mathcal{R}^r_{12}$ *are equivalent systems over which the structural rules are admissible.*

Proof Obviously $\mathcal{R}^r_{12}$ is a subsystem of $\mathcal{R}^{rl}_{12}$. The converse holds by the previous Theorem and the equivalence of the equality rules over systems containing $\{\ \overline{\text{Ref}},\ \text{Cut},\ \text{LC}\}$ established in Proposition 2. □

5 Restricting the Equality Rules

Theorem 5 can be strengthened by requiring that, when the context formula is an equality, the rules Rep^r_1 and Rep^r_2 change only its right-hand side. Let $\text{Rep}^{r=r}_1$ and $\text{Rep}^{r=r}_2$ be the restrictions of Rep^r_1 and Rep^r_2 obtained in that way.

Theorem 7. *The system* $\mathcal{R}_{12}^{r=r} = \{\overline{\text{Ref}}, \text{Rep}_1^{r=r}, \text{Rep}_2^{r=r}\}$ *is equivalent to* $\mathcal{R}_{12}^{r}$*, hence the structural rules are admissible in* $\mathbf{G3[mic]}^{\mathcal{R}_{12}^{r=r}}$*.*

Proof It suffices to show that if a sequent of the form $\Gamma \Rightarrow p = q$ is derivable in $\mathcal{R}_{12}^{r}$, then it is derivable in $\mathcal{R}_{12}^{r=r}$ as well. Given a derivation $\mathcal{D}$ in $\mathcal{R}_{12}^{r}$ of $\Gamma \Rightarrow p = q$ we proceed by induction on the number of Rep_1^r or Rep_2^r -inferences that act on an equality but are not $\text{Rep}_1^{r=r}$ or $\text{Rep}_2^{r=r}$ -inferences, to be called *undesired* inferences. If there are none we are done. Otherwise we select the topmost one call it J. Let us assume that it is of the form:

$$\frac{r = s, \Gamma^- \Rightarrow p'^{\circ}[x/r] = q'}{r = s, \Gamma^- \Rightarrow p'^{\circ}[x/s] = q'} \quad \text{Rep}_1^r$$

Since an initial sequent of the form $t = t',\ \Gamma \Rightarrow t = t'$ is derivable from $t = t', \Gamma \Rightarrow t = t$ by means of a $\text{Rep}_1^{r=r}$-inference, we may assume that the initial sequent of $\mathcal{D}$ has the form

$$r = s, \Gamma^- \Rightarrow p'^{\circ}[x/r] = p'^{\circ}[x/r]$$

If we replace the initial sequent of $\mathcal{D}$ by:

$$\frac{r = s, \Gamma^- \Rightarrow p'^{\circ}[x/s] = p'^{\circ}[x/s]}{r = s, \Gamma^- \Rightarrow p'^{\circ}[x/s] = p'^{\circ}[x/r]} \quad \text{Rep}_2^{r=r}$$

and the successive left-hand sides $p'^{\circ}[x/r]$ of the right equalities of $\mathcal{D}$ down to the premiss of J by $p'^{\circ}[x/s]$ we obtain the conclusion of J that therefore can be eliminated from the given derivation of $\Gamma \Rightarrow p = q$ thus obtaining a derivation that has one less undesired inference than $\mathcal{D}$. If J is an Rep_2^r the argument is the same except that the new initial inference is a $\text{Rep}_1^{r=r}$-inference rather than a $\text{Rep}_2^{r=r}$-inference. □

Let $\text{Rep}_1^{r/=}$ and $\text{Rep}_2^{r/=}$ be the rules $\text{Rep}_1^{r=r}$ and $\text{Rep}_2^{r=r}$ restricted to context formulae that are equalities and $\text{Rep}_1^{l/(=)}$ and $\text{Rep}_2^{l/(=)}$ be the rules Rep_1^l and Rep_2^l restricted to context formulae that are not equalities.

Theorem 8. *Let* $\mathcal{R}_{12}^{r/=,l/(=)}$ *be* $\{\overline{\text{Ref}},\ \text{Rep}_1^{r/=}, \text{Rep}_2^{r/=}, \text{Rep}_1^{l/(=)}, \text{Rep}_2^{l/(=)}\}$.

$\mathcal{R}_{12}^{r/=,l/(=)}$ *is equivalent to* $\mathcal{R}_{12}^{r}$*, therefore the structural rules are admissible in* $\mathbf{G3[mic]}^{\mathcal{R}_{l/(=)}^{r/=}}$*.*

Proof By Corollary 6 every sequent derivable in $\mathcal{R}_{12}^{r/=,l/(=)}$ is derivable in $\mathcal{R}_{12}^{r}$ as well. For the converse we note that if $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{R}_{12}^{r}$, then there is a formula A in Δ such that $\Gamma \Rightarrow A$ is also derivable in that system. If A is an equality, then the derivation of $\Gamma \Rightarrow A$ can use only $\mathrm{Rep}_1^{r/=}$ and $\mathrm{Rep}_2^{r/=}$, so that it belongs to $\mathcal{R}_{12}^{r/=,l/(=)}$. If A is not an equality we proceed by induction on the height of the derivation $\mathcal{D}$ in $\mathcal{R}_{12}^{r}$ of $\Gamma \Rightarrow A$ to show that it can be transformed into a derivation (of the same height) in $\mathcal{R}_{12}^{r/=,l/(=)}$. If $h(\mathcal{D}) = 0$, then $\Gamma \Rightarrow A$ is an initial sequent and the conclusion is obvious. If $h(\mathcal{D}) = n + 1$, then $\mathcal{D}$ ends either with an Rep_1^{r}-inference or with an Rep_2^{r}-inference. Let us assume, for example, that $\mathcal{D}$ ends with a Rep_1^{r}-inference. Then $\mathcal{D}$ has the form:

$$\dfrac{\begin{array}{c} P_1, \Gamma_1 \Rightarrow P_1 \\ \mathcal{D}_0 \\ r = s, \Gamma_n \Rightarrow P_n[x/r] \end{array}}{r = s, \Gamma_n \Rightarrow P_n[x/s]}$$

By induction hypothesis there is a derivation $\mathcal{D}_0'$ in $\mathcal{R}_{12}^{r/=,l/(=)}$ (of height n) of $r = s, \Gamma_n \Rightarrow P_n[x/r]$. $\mathcal{D}_0'$ has the form:

$$\begin{array}{c} \dfrac{r = s, P_n[x/r], \Lambda \Rightarrow P_n[x/r]}{r = s, \Lambda' \Rightarrow P_n[x/r]} \\ \vdots \\ r = s, \Gamma_n \Rightarrow P_n[x/r] \end{array}$$

In fact $\mathrm{Rep}_1^{l/(=)}$ and $\mathrm{Rep}_2^{l/(=)}$ do not introduce any new equality in their conclusion, so that all the equalities in the endsequent of $\mathcal{D}_0'$, in particular $r = s$, are present in the antecedent of every sequent in $\mathcal{D}_0'$. If we replace all the occurrences of $P_n[x/r]$ in the succedents of the sequents of $\mathcal{D}_0$ by $P[x/s]$ and introduce an initial $\mathrm{Rep}_2^{l/(=)}$-inference replacing s by r in $P_n[x/r]$ we obtain the desired derivation $\mathcal{D}'$ in $\mathcal{R}_{12}^{r/=,l/(=)}$ (of height $n + 1$), namely:

$$\begin{array}{c} \dfrac{\dfrac{r = s, P_n[x/s], \Lambda \Rightarrow P_n[x/s]}{r = s, P_n[x/r], \Lambda \Rightarrow P_n[x/s]}}{r = s, \Lambda' \Rightarrow P_n[x/s]} \\ \vdots \\ r = s, \Gamma_n \Rightarrow P_n[x/s] \end{array}$$

□

Clearly the proof goes through without any change if $\text{Rep}_1^{r/=}$ and $\text{Rep}_2^{r/=}$ are restricted to $\text{Rep}_1^{r/=_r}$ and $\text{Rep}_2^{r/=_r}$ that change only the right-hand side of the equality that they transform.

Thus, letting $\mathcal{R}_{12}^{r/=_r,l/(=)} = \{\overline{\text{Ref}}, \text{Rep}_1{}^{l/(=)}, \text{Rep}_2{}^{l/(=)}, \text{Rep}_1^{r/=_r}, \text{Rep}_2{}^{r/=_r}\}$, we have the following stregthened form of the previous Theorem:

Theorem 9. *$\mathcal{R}_{12}^{r/=_r,l/(=)}$ is equivalent to $\mathcal{R}_{12}^r$, therefore the structural rules are admissible in $\mathbf{G3}^{\mathcal{R}_{12}^{r/=_r,l/(=)}}$*

Interpreted in terms of the alternate tableau system in [8], pg. 294 where a branch can be closed if the negation of an identity $\neg t = t$ appears on it, and left-right and right-left replacement of equals can be applied to atomic formulae and to negation of equalities, this result, in the classical case, means that, strictness can be imposed (i.e. the formulae in which a replacement is performed are checked out) and the replacement rule can be applied only to atomic formula that are not equalities and to the right-hand side of negation of equalities. Note that in the same framework, by Theorem 5 strictness can be imposed and replacement of equals can be limited to negations of atomic formulae, whether equalities or not.

5.1 Orienting replacement in languages without function symbols

We prove that for languages without function symbols the structural rules are admissible in $\mathcal{R}_2^{rl} = \{\overline{Ref}, \text{Rep}_2^r, \text{Rep}_2^r\}$ by showing that for such languages $\mathcal{R}_2^{rl}$ is in fact equivalent to $\mathcal{R}_{12}^r$. The same holds, with the same proofs, for $\mathcal{R}_1^{rl}$.

Notation In the following a, b, c will stand for constants or free variables and $a \approx b$ may denote either one of $a = b$ or $b = a$.

Definition 10. *A chain of equalities connecting a and b denoted by $\gamma(a, b)$ is a multiset of equalities that can be arranged into a sequence of the form $a \approx a_1, a_1 \approx a_2, \ldots, a_{n-1} \approx b$. The empty set is a chain that connects any term with itself.*

Lemma 11. *Given a chain $\gamma(a, b)$ and an atomic formula A with at most one occurrence of x*

a) $\gamma(a, b) \Rightarrow a = b$ is derivable in $\mathcal{R}_2^{rl}$

b) $A[x/a], \gamma(a, b) \Rightarrow A[x/b]$ is derivable in $\mathcal{R}_2^{rl}$

Proof In both cases we proceed by induction on the length n of $a \approx a_1, a_1 \approx a_2, \ldots, a_{n-2} \approx a_{n-1}, a_{n-1} \approx b$.

a) For $n = 0$, $\gamma(a,b) = \emptyset$ and $a \equiv b$ so that $\gamma(a,b) \Rightarrow a = b$ is the instance $\Rightarrow a = a$ of $\overline{\text{Ref}}$. For $n = 1$, $\gamma(a,b)$ is either $a = b$ or $b = a$. In the former case $\gamma(a,b) \Rightarrow a = b$ is the initial sequent $a = b \Rightarrow a = b$, while in the latter case it has the following derivation in $\mathcal{R}_2^{rl}$:

$$\frac{b = a \Rightarrow a = a}{b = a \Rightarrow a = b}\ \text{Rep}_2^r$$

Assume $n > 1$. If $a_{n-1} \approx b$ is $a_{n-1} = b$, by induction hypothesis:

$$a \approx a_1, \ldots, a_{n-2} \approx b \Rightarrow a = b$$

has a derivation in $\mathcal{R}_2^{rl}$ from which we obtain the desired derivation in $\mathcal{R}_2^{rl}$ by the admissibility of LW that allows for the introduction of $a_{n-1} = b$ and a Rep_2^l-inference using $a_{n-1} = b$ as operating equality, namely:

$$\cfrac{\cfrac{a \approx a_1, \ldots, a_{n-2} \approx b \Rightarrow a = b}{a \approx a_1, \ldots, a_{n-2} \approx b,\ a_{n-1} = b \Rightarrow a = b}\ \text{LW}}{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1},\ a_{n-1} = b \Rightarrow a = b}\ \text{Rep}_2^l$$

If $a_{n-1} \approx b$ is $b = a_{n-1}$, by induction hypothesis:

$$a \approx a_1, \ldots, a_{n-2} \approx a_{n-1} \Rightarrow a = a_{n-1}$$

has a derivation $\mathcal{D}$ in $\mathcal{R}_2^{rl}$ from which we obtain the desired derivation in $\mathcal{R}_2^{rl}$ by the admissibility of LW that allows for the introduction of $b = a_{n-1}$ and a Rep_2^r-inference using $b = a_{n-1}$, namely:

$$\cfrac{\cfrac{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1} \Rightarrow a = a_{n-1}}{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1},\ b = a_{n-1} \Rightarrow a = a_{n-1}}\ \text{LW}}{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1},\ b = a_{n-1} \Rightarrow a = b}\ \text{Rep}_2^r$$

b) For $n = 0$, $A[x/a], \gamma(a,b) \Rightarrow A[x/b])$ reduces to the initial sequent $A[x/a] \Rightarrow A[x/a]$. For $n = 1$ we have the following derivations, depending on whether $a \approx b$ is $a = b$ or $b = a$:

$$\frac{A[x/b],\ a = b \Rightarrow A[x/b]}{A[x/a],\ a = b \Rightarrow A[x/b]}\ \text{Rep}_2^l \qquad \frac{A[x/a], b = a \Rightarrow A[x/a]}{A[x/a],\ b = a \Rightarrow A[x/b]}\ \text{Rep}_2^r$$

For $n > 1$ the argument is similar to that in *a*). If $a_{n-1} \approx b$ is $a_{n-1} = b$, we note that by induction hypothesis we have a derivation in $\mathcal{R}_2^{rl}$ of

$$A[x/a], a \approx a_1, \ldots, a_{n-2} \approx b \Rightarrow A[x/b]$$

from which the desired derivation is obtained by a weakening introducing $a_{n-1} = b$ followed by a Rep_2^l-inference transforming $a_{n-2} \approx b$ into $a_{n-2} \approx a_{n-1}$.

If $a_{n-1} \approx b$ is $b = a_{n-1}$, by induction hypothesis we have a derivation in $\mathcal{R}_2^{rl}$ of

$$A[x/a], a \approx a_1, \ldots, a_{n-2} \approx a_{n-1} \Rightarrow A[x/a_{n-1}]$$

from which the desired derivation is obtained by a weakening introducing $b = a_{n-1}$ and a Rep_2^r-inference transforming $A[x/a_{n-1}]$ into $A[x/b]$. □

Lemma 12. *Given an atomic formula A, m variables $x_1, \ldots, x_m$ having at most one occurrence in A and m chains $\gamma_1(a_1, b_1), \ldots, \gamma_m(a_m, b_m)$ the sequent:*

$$A[x_1/a_1, \ldots, x_m/a_m], \gamma_1(a_1, b_1), \ldots, \gamma_m(a_m, b_m) \Rightarrow A[x_1/b_1, \ldots, x_m/b_m]$$

is derivable in $\mathcal{R}_2^{rl}$.

Proof We proceed by a principal induction on m and a secondary induction on the length of $\gamma_m(a_m, b_m)$. For $m = 1$ the claim reduces to the previous lemma part b). Assuming it holds for $m - 1$ we have

$$1)\quad A[x_1/a_1, \ldots, x_{m-1}/a_{m-1}, x_m/a_m), \gamma_1(a_1, b_1), \ldots, \gamma_{m-1}(a_{m-1}, b_{m-1}) \Rightarrow \\ \Rightarrow A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/a_m]$$

as well as

$$2)\quad A[x_1/a_1, \ldots, x_{m-1}/a_{m-1}, x_m/b_m], \gamma_1(a_1, b_1), \ldots, \gamma_{m-1}(a_{m-1}, b_{m-1}) \Rightarrow \\ \Rightarrow A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/b_m]$$

are derivable in $\mathcal{R}_2^{rl}$. Then we can proceed by induction on the length l of $\gamma_m(a_m, b_m)$ to show that also

$$A[x_1/a_1, \ldots, x_m/a_m], \gamma_1(a_1, b_1), \ldots, \gamma_m(a_m, b_m) \Rightarrow A[x_1/b_1, \ldots, x_m/b_m]$$

is derivable in $\mathcal{R}_2^{rl}$. If $l = 0$ then $a_m \equiv b_m$ and the conclusion is immediate. If $l = 1$ then $\gamma_m(a_m, b_m)$ is either $a_m = b_m$ or $b_m = a_m$. In the first case we weaken the sequent 2) by adding $a_m = b_m$ and then apply a Rep_2^l-inference to tranform $A[x_1/a_1, \ldots, x_{m-1}/a_{m-1}, x_m/b_m]$ in the antecedent of 2) into $A[x_1/a_1, \ldots, x_{m-1}, x_m/a_m]$. Similarly if $\gamma_m(a_m, b_m)$ is $b_m = a_m$, we add $b_m = a_m$ to the antecedent of 1) and then apply a Rep_2^r-inference to transform $A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/a_m]$ in the consequent of 2) into $A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/b_m]$. For $l > 1$ let $\gamma(a_m, b_m)$ be $a_m \approx a_m^1, a_m^1 \approx a_m^2, \ldots, a_m^{l-2} \approx a_m^{l-1}, a_m^{l-1} \approx b_m$. If $a_m^{l-1} \approx b_m$ is $b_m = a_m^{l-1}$ we note that by induction hypothesis:

$$A[x_1/a_1, \ldots, x_{m-1}/a_{m-1}, x_m/a_m], \gamma_1(a_1, b_1), \ldots, \gamma_{m-1}(a_{m-1}, b_{m-1}),$$
$$a_m \approx a_m^1, a_m^1 \approx a_m^2, \ldots, a_m^{l-2} \approx a_m^{l-1} \Rightarrow A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/a_m^{l-1}]$$

is derivable in $\mathcal{R}_2^{rl}$. Then it suffices to weaken the antecedent by adding $b_m = a_m^{l-1}$ and apply a Rep_2^r-inference to transform $A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/a_m^{l-1}]$ into $A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/b_m]$ to obtain the desired derivation in $\mathcal{R}_2^{rl}$ of

$$*) \quad A[x_1/a_1, \ldots, x_{m-1}/a_{m-1}, x_m/a_m], \gamma_1(a_1, b_1), \ldots, \gamma_{m-1}(a_{m-1}, b_{m-1}),$$
$$a_m \approx a_m^1, a_m^1 \approx a_m^2, \ldots, a_m^{l-2} \approx a_m^{l-1}, b_m = a_m^{l-1}$$
$$\Rightarrow A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/b_m]$$

On the other hand if $a_m^{l-1} \approx b_m$ is $a_m^{l-1} = b_m$ we note that by induction hypothesis there is a derivation in $\mathcal{R}_2^{rl}$ of

$$A[x_1/a_1, \ldots, x_{m-1}/a_{m-1}, x_m/a_m], \gamma_1(a_1, b_1), \ldots, \gamma_{m-1},$$
$$a_m \approx a_m^1, a_m^1 \approx a_m^2, \ldots, a_m^{l-2} \approx b_m$$
$$\Rightarrow A[x_1/b_1, \ldots, x_{m-1}/b_{m-1}, x_m/b_m]$$

that can be weakened by the addition of $a_m^{l-1} = b_m$ in the antecedent to be used to transform, by means of a Rep_2^l-inference, $a_m^{l-2} \approx b_m$ into $a_m^{l-2} \approx a_m^{l-1}$ in order to obtain a derivation of $*)$ in $\mathcal{R}_2^{rl}$. □

Lemma 13. *a) If $\Gamma \Rightarrow a = b$ is derivable in $\mathcal{R}_{12}^r$, then Γ includes a chain $\gamma(a, b)$.*

b) If A is not an equality and $\Gamma \Rightarrow A$ is derivable in $\mathcal{R}_{12}^r$, then for some m there are two m-tuples $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, such that A has the form $A^\circ[x_1/b_1, \ldots, x_m/b_m]$ and Γ contains $A^\circ[x_1/a_1, \ldots, x_m/a_m]$ as well as m chains $\gamma_1(a_1, b_1), \ldots, \gamma_m(a_m, b_m)$.

Proof By Theorem 7 we can proceed by induction on the height of a derivation $\mathcal{D}$ in $\mathcal{R}_{12}^{r=_r}$ of $\Gamma \Rightarrow a = b$ or $\Gamma \Rightarrow A$.

a) If $h(\mathcal{D}) = 0$ then $\Gamma \Rightarrow a = b$ is an instance of $\overline{\text{Ref}}$ i.e. $a \equiv b$ and we can let $\gamma(a, b) = \emptyset$ or it is an initial sequent, i.e. $a = b$ occurs in Γ and we can let $\gamma(a, b) = \{a = b\}$.

If $h(\mathcal{D}) > 0$ and $\mathcal{D}$ ends with a $\text{Rep}_1^{r=_r}$-inference, i.e it is of the form:

$$\cfrac{\begin{matrix}\mathcal{D}_0\\ a = b, \Gamma^- \Rightarrow c = a\end{matrix}}{a = b, \Gamma^- \Rightarrow c = b}$$

by induction hypothesis we have that $a = b, \Gamma^-$ is of the form $\gamma'(c, a), \Gamma^{--}$. If $a \approx b$ does not belong to $\gamma'(c, a)$ it suffices to let $\gamma(a, b) = \gamma'(c, a) \cup \{a = b\}$. Otherwise, since $\gamma'(c, a)$ can be represented as

$$c \approx a_1, \ldots, a_i \approx a, a \approx b, b \approx a_{i+3}, \ldots, a_{n-1} \approx a$$

we can let $\gamma(c, b) = \{c \approx a_1, \ldots, a_i \approx a, a \approx b\}$. The same argument applies if $\mathcal{D}$ ends with a $\mathrm{Rep}_2^{r=r}$-inference.

b) If $h(\mathcal{D}) = 0$ then A occurs in Γ and the claim holds with $m = 0$.

If $h(\mathcal{D}) > 0$ and $\mathcal{D}$ ends with a Rep_1^r-inference, assuming, for the sake of notational simplicity, that the induction hypothesis holds with $m' = 2$, the last inference of $\mathcal{D}$ has one of the following three forms:

$$i) \qquad \frac{b_1 = b,\ A^\circ[x_1/a_1, x_2/a_2], \gamma_1'(a_1, b_1), \gamma_2'(a_2, b_2), \Gamma^- \Rightarrow A^\circ[x_1/b_1, x_2/b_2]}{b_1 = b,\ A^\circ[x_1/a_1, x_2/a_2], \gamma_1'(a_1, b_1), \gamma_2'(a_2, b_2), \Gamma^- \Rightarrow A^\circ[x_1/b, x_2/b_2]}$$

$$ii) \qquad \frac{b_2 = b,\ A^\circ[x_1/a_1, x_2/a_2], \gamma_1'(a_1, b_1), \gamma_2'(a_2, b_2), \Gamma^- \Rightarrow A^\circ[x_1/b_1, x_2/b_2]}{b_2 = b,\ A^\circ[x_1/a_1, x_2/a_2], \gamma_1'(a_1, b_1), \gamma_2'(a_2, b_2), \Gamma^- \Rightarrow A^\circ[x_1/b_1, x_2/b]}$$

$$iii) \quad \frac{a = b, A^\circ[x_1/a_1, x_2/a_2, x/a], \gamma_1'(a_1, b_1), \gamma_2'(a_2, b_2), \Gamma^- \Rightarrow A^\circ[x_1/b_1, x_2/b_2, x/a]}{a = b, A^\circ[x_1/a_1, x_2/a_2, x/a], \gamma_1'(a_1, b_1), \gamma_2'(a_2, b_2), \Gamma^- \Rightarrow A^\circ[x_1/b_1, x_2/b_2, x/b]}$$

In case *i*), if $b_1 \approx b$ does not belong to $\gamma_1'(a_1, b_1)$ it suffices to let $\gamma_1(a_1, b) = \gamma'(a_1, b_1) \cup \{b_1 = b\}$ while if $b_1 \approx b$ does belong to $\gamma_1'(a_1, b_1)$, as in the similar case concerning *a*), we have that $\gamma_1'(a_1, b_1)$ already contains a chain connecting a_1 and b that can be taken as $\gamma_1(a_1, b)$. In both cases we let $\gamma_2 = \gamma_2'$ so that $m = m'$.

Case *ii*) is entirely similar to Case *i*).

Finally in Case *iii*) it suffices to let $\gamma_1 = \gamma_1'$, $\gamma_2 = \gamma_2'$ and $\gamma_3 = \{a = b\}$ so that $m = 3$. □

As an immediate consequence of the two previous lemmas and the admissibility of left weakening we have the following:

Proposition 14. *For languages without function symbols, a sequent derivable in* $\mathcal{R}_{12}^r$ *is derivable also in* $\mathcal{R}_2^{rl}$.

Theorem 15. *For languages without function symbols,* $\mathcal{R}_2^{rl}$ *is equivalent to* $\mathcal{R}_{12}^r$, *hence the structural rules are admissible in* $\mathbf{G3[mic]}^{\mathcal{R}_2^{rl}}$.

Proof By Corollary 6 $\mathcal{R}_2^{rl}$ is a subsystem of $\mathcal{R}_{12}^r$ and the converse holds by the previous Proposition. □

In the classical case, interpreted in terms of the tableau system introduced in [9], which deals with languages without function symbols, this results is a remarkable improvement of the result in 5.1 of [9], since it means that not only strictness can be required, in contrast with the author's emphasized warning "*Don't check either premiss*" (pg.77), but also that replacement can be restricted to left-right replacement.

5.2 Orienting replacement in languages with function symbols

Let Rep_1^{l+} and Rep_2^{l+} be the rules Rep_1^l and Rep_2^l whose instances concerning equalities (E) are replaced by:

$$\frac{s = r, E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{s = r, E[x/s], \Gamma \Rightarrow \Delta} \qquad \text{and} \qquad \frac{r = s,\ E[x/s], E[x/r], \Gamma \Rightarrow \Delta}{r = s,\ E[x/s], \Gamma \Rightarrow \Delta}$$

respectively.

Note that, thanks to the admissibility of left weakening, Rep_1^{l+} and Rep_2^{l+} are strengthening of Rep_1^l and Rep_2^l respectively. On the other hand, it is straightforward that Proposition 2 extends to such rules as well.

Proposition 16. *The rule* Rep_1^r *is admissible in* $\mathcal{R}_2^{rl+} = \{\overline{\mathrm{Ref}}, \mathrm{Rep}_2^{l+},\ \mathrm{Rep}_2^r\}$. *The same holds with* 1 *and* 2 *exchanged.*

Proof We may assume that all the rules under consideration replace exactly one occurrence of a term by another (see [12] and [14]). Then we proceed by induction on the height $h(\mathcal{D})$ of a derivation $\mathcal{D}$ in $\{\overline{\mathrm{Ref}}, \mathrm{Rep}_1^r, \mathrm{Rep}_2^{l+},\ \mathrm{Rep}_2^r\}$ that ends with an Rep_1^r-inference and contains no other Rep_1^r-inference, to show that $\mathcal{D}$ can be transformed into a derivation $\mathcal{D}'$ in $\mathcal{R}_2^{rl+}$ of the same endsequent. If $h(\mathcal{D}) = 1$, then $\mathcal{D}$ has the form:

$$\frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]}$$

where $r = s, \Gamma \Rightarrow \Delta, P[x/r]$ is either an initial sequent or an instance of $\overline{\mathrm{Ref}}$. Case 1. $r = s, \Gamma \Rightarrow \Delta, P[x/r]$ is an initial sequent. Then we have the following subcases:

Case 1.1. $(r = s, \Gamma) \cap \Delta \neq \emptyset$, then $r = s, \Gamma \Rightarrow \Delta, P[x/s]$ is also an initial sequent.

Case 1.2. $r = s, \Gamma \Rightarrow \Delta, P[x/r]$ is of the form $r = s,\ P[x/r],\ \Gamma' \Rightarrow \Delta, P[x/r]$. Then $\mathcal{D}$ can be transformed into:

$$\dfrac{r = s,\ P[x/s], \Gamma' \Rightarrow \Delta, P[x/s]}{r = s,\ P[x/r], \Gamma' \Rightarrow \Delta, P[x/s]}\ \text{Rep}_2^l$$

Case 1.3. $r = s, \Gamma \Rightarrow \Delta, P[x/r]$ is of the form $r = s, \Gamma \Rightarrow \Delta, r = s$.

Case 1.3.1.$P \equiv x = s$, hence $\mathcal{D}$ has the form:

$$\dfrac{r = s,\ \Gamma \Rightarrow \Delta, (x = s)[x/r]}{r = s,\ \ \Gamma \Rightarrow \Delta, (x = s)[x/s]}$$

then the conclusion of $\mathcal{D}$ is an instance of $\overline{\text{Ref}}$, that can be taken as $\mathcal{D}'$.

Case 1.3.2. $P \equiv s^\circ$, with $s^\circ[x/r] \equiv s$, hence $\mathcal{D}$ has the form:

$$\dfrac{r = s^\circ[x/r], \Gamma \Rightarrow \Delta,\ r = s^\circ[x/r]}{r = s^\circ[x/r], \Gamma \Rightarrow \Delta,\ r = s^\circ[x/s^\circ[x/r]]}$$

Then $\mathcal{D}$ can be transformed into:

$$\dfrac{\dfrac{r = s^\circ[x/r],\ \Gamma \Rightarrow \Delta,\ s^\circ[x/s^\circ[x/r]] = s^\circ[x/s^\circ[x/r]]}{r = s^\circ[x/r],\ \Gamma \Rightarrow \Delta,\ s^\circ[x/r] = s^\circ[x/s^\circ[x/r]]}\ \text{Rep}_2^r}{r = s^\circ[x/r],\ \Gamma \Rightarrow \Delta,\ r = s^\circ[x/s^\circ[x/r]]}\ \text{Rep}_2^r$$

Case 2. $r = s, \Gamma \Rightarrow \Delta, P[x/r]$ is an instance of $\overline{\text{Ref}}$. Then we have the following subcases:

Case 2.1. The principal formula is in Δ. Then $r = s, \Gamma \Rightarrow \Delta, P[x/s]$ is also an instance of $\overline{\text{Ref}}$.

Case 2.2. The principal formula is $P[x/r]$. Then $P[x/r]$ has the form $t = t$, hence P has the form $t^\circ = t$, or $t = t^\circ$ with $t \equiv t^\circ[x/r]$

Case 2.2.1. $P \equiv t^\circ = t$. Then $\mathcal{D}$ is transformed into:

$$\dfrac{r = s, \Gamma \Rightarrow \Delta, t^\circ[x/s] = t^\circ[x/s]}{r = s, \Gamma \Rightarrow \Delta, t^\circ[x/s] = t}\ \text{Rep}_2^r$$

Case 2.2.2. $P \equiv t = t^\circ$. Then $\mathcal{D}$ is transformed into:

$$\dfrac{r = s, \Gamma \Rightarrow \Delta, t^\circ[x/s] = t^\circ[x/s]}{r = s, \Gamma \Rightarrow \Delta, t = t^\circ[x/s]}\ \text{Rep}_2^r$$

If $h(\mathcal{D}) > 0$ we distinguish the following cases:

Case 3. The last inference of the immediate subderivation of $\mathcal{D}$ is an Rep_2^r-inference.

Case 3.1.

$$\dfrac{\dfrac{q = p,\ r = s,\ \Gamma \Rightarrow \Delta',\ Q[y/p],\ P[x/r]}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta',\ Q[y/q],\ P[x/r]}}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta',\ Q[y/q],\ P[x/s]} \quad \text{Rep}_2^r$$

is transformed into:

$$\dfrac{\dfrac{q = p,\ r = s,\ \Gamma \Rightarrow \Delta',\ Q[y/p],\ P[x/r]}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta',\ Q[y/p],\ P[x/s]}}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta',\ Q[y/q],\ P[x/s]} \quad \begin{matrix}\text{ind}\\ \text{Rep}_2^r\end{matrix}$$

Case 3.2.

$$\dfrac{\dfrac{q = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[y/p, x/r]}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta\ P[y/q, x/r]}}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[y/q, x/s]} \quad \text{Rep}_2^r$$

is transformed into:

$$\dfrac{\dfrac{q = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[y/p, x/r]}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta\ P[y/p, x/s]}}{q = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[y/q, x/s]} \quad \begin{matrix}\text{ind}\\ \text{Rep}_2^r\end{matrix}$$

Case 3.3.

$$\dfrac{\dfrac{q^\circ[y/r] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/p]}{q^\circ[y/r] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/q^\circ[y/r]]}}{q^\circ[y/r] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/q^\circ[y/s]]} \quad \text{Rep}_2^r$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{q^\circ[y/r] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/p]}{q^\circ[y/r] = p,\ q^\circ[y/s] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/p]}}{q^\circ[y/r] = p,\ q^\circ[y/s] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/q^\circ[y/s]]}}{q^\circ[y/r] = p,\ r = s,\ \Gamma \Rightarrow \Delta,\ P[x/q^\circ[y/s]]} \quad \begin{matrix}\text{LW}\\ \text{Rep}_2^r\\ \text{Rep}_2^{l+}\end{matrix}$$

Case 3.4.

$$\dfrac{\dfrac{q=p,\ r^{\circ}[x/q]=s,\ \Gamma\Rightarrow\Delta,P[x/r^{\circ}[y/p]]}{q=p,\ r^{\circ}[x/q]=s,\ \Gamma\Rightarrow\Delta,P[x/r^{\circ}[y/q]]}}{q=p,\ r^{\circ}[x/q]=s,\ \Gamma\Rightarrow\Delta,P[x/s]}\ \text{Rep}^r_2$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{q=p,\ r^{\circ}[x/q]=s,\ \Gamma\Rightarrow\Delta,P[x/r^{\circ}[y/p]]}{q=p,\ r^{\circ}[x/q]=s,\ r^{\circ}[x/p]=s,\ \Gamma\Rightarrow\Delta,P[x/r^{\circ}[y/p]]}\ \text{LW}}{q=p,\ r^{\circ}[x/q]=s,\ r^{\circ}[x/p]=s,\ \Gamma\Rightarrow\Delta,P[x/s]}\ \text{ind}}{q=p,\ r^{\circ}[x/q]=s,\ \Gamma\Rightarrow\Delta,P[x/s]}\ \text{Rep}^{l+}_2$$

Case 4. The last inference of the immediate subderivation of $\mathcal{D}$ is an Rep^{l+}_2-inference acting on a formula Q that is not an equality, namely an Rep^{l}_2-inference.

$$\dfrac{\dfrac{q=p,\ r=s,Q[y/p],\Gamma'\Rightarrow\Delta,P[x/r]}{q=p,\ r=s,Q[y/q],\Gamma'\Rightarrow\Delta,P[x/r]}}{q=p,\ r=s,\ Q[y/q],\Gamma'\Rightarrow\Delta,P[x/s]}\ \text{Rep}^l_2$$

is transformed into:

$$\dfrac{\dfrac{q=p,r=s,Q[y/p],\Gamma'\Rightarrow\Delta,P[x/r]}{q=p,r=s,Q[y/p],\Gamma'\Rightarrow\Delta,P[x/s]}\ \text{ind}}{q=p,r=s,Q[y/q],\Gamma'\Rightarrow\Delta,P[x/s]}\ \text{Rep}^l_2$$

Case 5. The last inference of the immediate subderivation of $\mathcal{D}$ is a Rep^{l+}_2-inference acting on an equality E. In this case we can proceed as in Case 4, by first inverting the last Rep^r_1- inference with the preceding Rep^{l+}_2-inference and then applying the induction hypothesis. □

Theorem 17. *The systems $\mathcal{R}^r_{12}$ and $\mathcal{R}^{rl+}_2$ are equivalent, hence the structural rules are admissible in* $\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}^{rl+}_2}$. *The same holds for $\mathcal{R}^{rl+}_1=\{\overline{\text{Ref}},\text{Rep}^{l+}_1,\ \text{Rep}^r_1\}$.*

Proof Since, by the previous Proposition, Rep^r_1 is admissible in $\mathcal{R}^{rl+}_2$, $\mathcal{R}^r_{12}$ is a subsystem of $\mathcal{R}^{rl+}_2$. By Theorem 5 and Proposition 2 we have the converse inclusion. □

Let $\mathcal{R}^{rl}_1$ and $\mathcal{R}^{rl}_2$ be $\{\overline{\text{Ref}},\text{Rep}^l_1,\ \text{Rep}^r_1\}$ and $\{\overline{\text{Ref}},\text{Rep}^l_2,\ \text{Rep}^r_2\}$ respectively.

Proposition 18. *$\mathcal{R}_1^{rl+}$ and $\mathcal{R}_2^{rl+}$ are equivalent to $\mathcal{R}_1^{rl} + \mathrm{LC}^=$ and $\mathcal{R}_2^{rl} + \mathrm{LC}^=$ respectively.*

Proof Rep_1^{l+} and Rep_2^{l+} are immediately derivable by means of $\mathrm{LC}^=$ from Rep_1^l and Rep_2^l respectively. On the other hand $\mathrm{LC}^=$ is admissible in both $\mathcal{R}_1^{rl+}$ and $\mathcal{R}_2^{rl+}$ by the previous Theorem. □

This naturally leads to what we consider a quite significant problem left open by our investigations:

Question Is it possible to extend Theorem 15 to languages endowed with function symbols, namely to replace $\mathcal{R}_2^{rl+}$ by $\mathcal{R}_2^{rl}$ in Theorem 17?

In the classical case, Proposition 17 yields that in the alternate tableau system in [8] pg.294 it is possible to impose strictness, except that on equalities (and γ-formulae of course), and at the same time restrict replacement to left-right replacement, provided it is allowed on all atomic and negation of atomic formulae. In that framework the above question amounts to asking whether strictness can be imposed also on equalities. It amounts also to asking whether in ($\approx$) in [6], namely:

$$\frac{r \approx s, \Gamma[x/r] \Rightarrow \Delta[x/r]}{r \approx s, \Gamma[x/s] \Rightarrow \Delta[x/s]}$$

$\approx$ can be replaced by $=$.

Remark In the semantic tableau method mentioned in the Introduction, in which one can add identities at will and replacement is restricted to left-right replacement on atomic formule, it follows from the result in [14], that strictness can be imposed on equalities as well.

6 Counterexamples

Since the weakening rules and the right contraction rule are admissible in all the systems consisting of $\overline{\mathrm{Ref}}$ and some of the equality rules, we will concentrate on the possible failure of the left contraction LC and/or the Cut rule. By Proposition 2 and Theorem 5, all the axioms and rules for equality that we have considered are admissible in $\mathcal{R}_{12}^r$. Thus, by Corollary 3, to show that at least one among LC and Cut is not admissible in a system $\mathcal{S}$ it suffices to find a sequent derivable in $\mathcal{R}_{12}^r$ but not in $\mathcal{S}$. A case of this kind in which LC is present, thus obviously admissible, and, therefore, Cut is not admissible, is provided by $\mathcal{S}_1 = \{\overline{\mathrm{Ref}}, \mathrm{LC}, \mathrm{Rep}_2^{l+}, \mathrm{Rep}_1^r\}$.

In fact for a, b and c distinct, the sequent $a = c, b = c \Rightarrow a = b$, which is derivable in $\mathcal{R}^r_{12}$, is not derivable in $\mathcal{S}_1$. As a matter of fact no sequent of the form

$$*) \quad a = c, \ldots a = c, b = c, \ldots, b = c, c = c, \ldots, c = c \Rightarrow a = b$$

is derivable in $\mathcal{S}_1$, since it can be the conclusion of LC, Rep^{l+}_2 or Rep^r_1-inference only if its premiss has already the form $*)$ and no initial sequent or instance of $\overline{\mathrm{Ref}}$ has that form. Clearly the same holds if in $\mathcal{S}_1$, Rep^{l+}_2 is replaced by the more extended rule Rep. A similar argument applies to $\mathcal{S}_2 = \{\overline{\mathrm{Ref}}, \mathrm{LC}, \mathrm{Rep}^{l+}_1, \mathrm{Rep}^r_2\}$ with respect to the sequent $c = b,\ c = a \Rightarrow a = b$ which is derivable in $\mathcal{R}^r_{12}$ but not in $\mathcal{S}_2$ and to the system obtained by replacing Rep^{l+}_1 by Rep'. While for the above systems it is the admissibility of Cut that fails, $\{\overline{\mathrm{Ref}}, \mathrm{Cut}, =_1, =_2\}$ is a system in which it is the admissibility of LC, actually of $\mathrm{LC}^=$, that fails, since, $a = f(a), a = f(a) \Rightarrow a = f(f(a))$ is derivable, but $a = f(a) \Rightarrow a = f(f(a))$ is not. Another example of the same sort is provided by $\{\overline{\mathrm{Ref}}, \mathrm{Cut}, \mathrm{CNG}\}$, which is easily seen to be equivalent to $\{\overline{\mathrm{Ref}}, \mathrm{Cut}, =_1, =_2\}$. Although in general it may happen for a rule not to be admissible in a system but admissible in a weaker one, for the system we are considering, since the failure of the admissibility of some of the structural rules is witnessed by the underivability of some sequent, which is obviously preserved by weakening a system, if they are not all admissible in $\mathcal{S}$ and $\mathcal{S}'$ is a subsystem of $\mathcal{S}$, then they are not all admissible in $\mathcal{S}'$ either. For example, since $\{\overline{\mathrm{Ref}}, \mathrm{CNG}\}$ is a subsystem $\{\overline{\mathrm{Ref}}, \mathrm{Cut}, \mathrm{CNG}\}$, LC and Cut are not both admissible also in $\{\overline{\mathrm{Ref}}, \mathrm{CNG}\}$. Actually that is still a case in which it is LC to be not admissible, since Cut remains admissible as it can be easily verified proceeding by induction on the height of the derivation in $\{\overline{\mathrm{Ref}}, \mathrm{CNG}\}$ of its second premiss. But note that, by 4) in Proposition 2 and the analogue for Rep^r_2 in the proof of Theorem 5, it suffices to add to $\{\overline{\mathrm{Ref}}, \mathrm{CNG}\}$ the left contraction rule restricted to equalities $\mathrm{LC}^=$ to obtain a system equivalent to $\mathcal{R}^r_{12}$ and, therefore, the admissibility of both LC and Cut.

7 Semishortening derivations

Let us recall from [12] the following definition:

Definition 19. *Let $\prec$ be any antisymmetric relation on terms. An application of an equality rule with input term r and output term s is said to be* nonlengthening *if $s \not\prec r$ and* shortening *if $r \prec s$. A derivation is said to be* semishortening *if all its equality inferences with index* 2 *are nonlengthening and those with index* 1 *are shortening.*

For languages without function symbols, the results in [12] apply to the present context as well. In fact an easy modification of the proof of Lemma 11 and Lemma 12 combined with Lemma 13 establishes the following result:

Proposition 20. *For languages without function symbols, if* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathcal{R}^r_{12}$*, then* $\Gamma \Rightarrow \Delta$ *has a semishortening derivation in* $\mathcal{R}^{rl}_{12}$*.*

We check in detail only part a) of the extension of Lemma 11 leaving the rest to the reader.

Lemma 21. *If* $\gamma(a, b)$ *is a chain, then* $\gamma(a, b) \Rightarrow a = b$ *has a semishortening derivation in* $\mathcal{R}^{rl}_{12}$*.*

Proof As in the proof of Lemma 11. assuming $\gamma(a, b)$ is $a \approx a_1$, $a_1 \approx a_2$, ..., $a_{n-2} \approx a_{n-1}$, $a_{n-1} \approx b$. we proceed by induction on its length n. For $n = 0$, the only change occurs if $\gamma(a, b) = \{b = a\}$. If so, we distinguish two cases displaying the corresponding semishortening derivation claimed to exist:

Case 1 $b \prec a$.

$$\frac{b = a \Rightarrow b = b}{b = a \Rightarrow a = b}\ \text{Rep}^r_1$$

Case 2 $b \not\prec a$.

$$\frac{b = a \Rightarrow a = a}{b = a \Rightarrow a = b}\ \text{Rep}^r_2$$

As for the induction step, let us assume that $|\gamma(a, b)| = n > 1$ so that $\gamma(a, b)$ is of the form:

$$a \approx a_1, \ldots, a_{n-2} \approx a_{n-1}, a_{n-1} \approx b$$

If $a_{n-1} \approx b$ is $a_{n-1} = b$ we distinguish two cases:

Case 1 $a_{n-1} \prec b$. By induction hypothesis in $\mathcal{R}^{rl}_{12}$ there is a semishortening derivation $\mathcal{D}$ of:

$$a \approx a_1, \ldots, a_{n-2} \approx a_{n-1} \Rightarrow a = a_{n-1}$$

Then the following is a semishortening derivation of $\gamma(a, b) \Rightarrow a = b$:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ a \approx a_1, \ldots, a_{n-2} \approx a_{n-1} \Rightarrow a = a_{n-1}\end{array}}{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1}, a_{n-1} = b \Rightarrow a = a_{n-1}}\ \text{LW}}{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1}, a_{n-1} = b \Rightarrow a = b}\ \text{Rep}^r_1$$

Case 2 $a_{n-1} \not\prec b$. By induction hypothesis in $\mathcal{R}^{rl}_{12}$ there is a semishortening derivation $\mathcal{D}$ of:

$$a \approx a_1, \ldots, a_{n-2} \approx b \Rightarrow a = b$$

Then the following is a semishortening derivation of $\gamma(a, b) \Rightarrow a = b$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ a \approx a_1, \ldots, a_{n-2} \approx b \Rightarrow a = b\end{array}}{a \approx a_1, \ldots, a_{n-2} \approx b, a_{n-1} = b \Rightarrow a = b}\,\text{LW}}{a \approx a_1, \ldots, a_{n-2} \approx a_{n-1}, a_{n-1} = b \Rightarrow a = b}\,\text{Rep}^r_2$$

If $a_{n-1} \approx b$ is $b = a_{n-1}$ the proof is analogous. □

On the other hand, for languages with function symbols, the result holds provided $\mathcal{R}^{rl}_{12}$ is strengthened into $\mathcal{R}^{rl+}_{12}$.

Proposition 22. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathcal{R}^r_{12}$*, then* $\Gamma \Rightarrow \Delta$ *has a semishortening derivation in* $\mathcal{R}^{rl+}_{12}$*.*

Proof It suffices to show that Rep^r_1 and Rep^r_2 are admissible in the calculus $\mathcal{R}^{rl+}_{12\prec}$, namely $\mathcal{R}^{rl+}_{12}$ with the applications of Rep^{l+}_1 and Rep^r_1 required to be shortening, denoted by $\text{Rep}^{l+}_{1\prec}$ and $\text{Rep}^r_{1\prec}$, and the applications of Rep^{l+}_2 and Rep^r_2 to be nonlengthening, denoted by $\text{Rep}^{l+}_{2\prec}$ and $\text{Rep}^r_{2\prec}$.

We proceed by induction on the height of a derivation in $\mathcal{R}^{rl+}_{12\prec}$ of the premiss of a non shortening Rep^r_1-inference or of a lenghtening Rep^r_2-inference.

As for a non shortening Rep^r_1-inference, if the derivation of the premiss is an initial sequent or an instance of $\overline{\text{Ref}}$ or ends with a $\text{Rep}^r_{2\prec}$ or a $\text{Rep}^{l+}_{2\prec}$ we apply the same transformations used in the proof of Proposition 16. Inspection of the various cases reveals that in the transformed derivation, the given non shortening Rep^r_1-inference is replaced by a Rep^l_2-inference that, having the same operating equality, turns out to be non lenghtening. Furthermore if the derivation of the premiss ends with a $\text{Rep}^r_{1\prec}$ or a $\text{Rep}^{l+}_{1\prec}$-inference we can perform similar tranformations leading to a derivation in $\mathcal{R}^{rl+}_{12\prec}$ of the conclusion. The case of a lengthening Rep^r_2-inference is dealt with in a similar way. We leave the details to the reader. □

Theorem 23. *The systems* $\mathcal{R}^r_{12}$ *and* $\mathcal{R}^{rl+}_{12\prec}$ *are equivalent, hence the structural rules are admissible in* $\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}^{rl+}_{12\prec}}$*.*

Proof By the previous Proposition, $\mathcal{R}^r_{12}$ is a subsystem of $\mathcal{R}^{rl+}_2$. The conclusion follows by Theorem 5 and Proposition 2 □

The proof of Proposition 22 uses the strengthened form $\text{Rep}_1^{l+}, \text{Rep}_2^{l+}$ of the rules $\text{Rep}_1^{l}, \text{Rep}_2^{l}$. However we have no counterexample, i.e. no particular $\prec$, showing that Proposition 22 does not hold for $\mathcal{R}_{12\prec}^{rl}$, in particular, according to the problem at the end of Section 7, since $\mathcal{R}_{12\emptyset}^{rl+}$ amounts to the same as $\mathcal{R}_{2}^{rl+}$, we do not have one for $\prec = \emptyset$.

Note In case $\prec$ is the relation induced by *rank*-comparison i.e. if $r \prec s$ if and only if the height (of the formation tree) of r is smaller than that of s, the derivability in $\mathcal{R}_{12\prec}^{rl}$ is closely related to the notion of a sequent being *directly demonstrable* as defined in [10], pg.90.

8 Conclusion

We have shown how the Gentzen's sequent calculi for first order logic with equality studied in [12] naturally evolve into their structural free counterparts based on Dragalin's multisuccedent calculi for minimal, intuitionistic and classical logic. We have shown that various restrictions limiting the scope of the replacement in the equality rules are possible. In the classical case all such results ensure the possibility of placing corresponding restrictions on the semantic tableau method for first order logic with equality. A particularly significant result is the possibility of imposing strictness as well as orientation of the replacement of equals in case the language lacks function symbols. On the way of extending this orientability with strictness result to general languages we have shown its reducibility to the admissibility of the Left Contraction Rule for equalities. Whether or not orientability can be obtained without adding such a contraction rule remains an open problem to be settled. Furthermore we have discussed to what extent the results in [12] concerning semishortening derivations can be extended to the present context leaving open a problem that includes the previous one as a particular case. While for languages with function symbols the question of the orientability of the replacement, the lack of which is a cause inefficiency of the tableau method for first order logic with equality, remains open, the refinements including strictness that we have provided naturally calls for their implementation. For that, to start with one can follow the lines of [8] which uses the free- variable tableau method to deal with the γ and δ- reductions. In that direction it would be interesting to investigate whether one can put more severe restrictions than the usual one adopted in [10] for the choice of terms to be used in the γ-expansions. Once such implementation are defined the comparison with other approaches such as those in [15], [4], [2] and [5] is naturally in order.

Of special interest should be the investigation of the result we have obtained in the intuitionistic case, expecially in connection with the treatment in [17]. The present work sets the theoretical ground for further more computationally oriented work along such lines.

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