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★**BCK algebras versus m-BCK algebras—foundations.**

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The author points out in this monograph that the commutative algebraic structures connected directly or indirectly with classical/non-classical logics belong to two parallel 'worlds'.

(1) The 'world' of *commutative algebras of logic* (called here *unital implicative-magmas*)

$$(A; \to, 1),$$

where there is (essentially) one implication, $\to$, and an element 1 verifying property (M):

$$1 \to x = x.$$

Here are some algebras belonging to this 'world': *Hilbert algebras*, *BCK algebras*, *BCI algebras*, *Wajsberg algebras*; several old generalizations of BCI or of BCK algebras, namely: *BCH algebras*, *BCC algebras* (also called *BIK*$^+$ *algebras*), *BZ algebras* (also called *weak-BCC algebras*), *BH algebras*, *BE algebras*, *L algebras*, *implicative-Boolean algebras*, structures with term-equivalent definition to Boolean algebras, *CI algebras*, and *pre-BCK algebras*; new generalizations of BCI or of BCK algebras introduced by the author, namely: *RM, pre-BZ, RME* (= *CI*), *RME**, pre-BCI, aRM* (= *BH*), *BCH** algebras*, and *RML, pre-BCC, aRML, BE**, aBE, aBE** algebras*, respectively, and many others; *M, ME, ML, MEL algebras*, *implicative-groups*, structures with term-equivalent definition to groups, and *implicative-ortholattices*, and structures with term-equivalent definition to ortholattices, all introduced by the author.

This 'world' is a framework centered on BCK algebras. The algebras $(A; \to, 1)$, verifying the basic property (M) are called *M algebras*; among the M algebras with additional operations, there are the algebras

$$(A; \to, 0, 1)$$

(where a negation can be defined by: $x^- \stackrel{\text{def}}{=} x \to 0$), or

$$(A; \to, {}^-, 1),$$

with $0 \stackrel{\text{def}}{=} 1^-$, where 1 is the *last element*, verifying (or not) (Ex) (Exchange):

$$x \to (y \to z) = y \to (x \to z);$$

and an internal binary relation can be defined by:

$$x \le y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad x \to y = 1$$

($\le$ can be a pre-order, an order, or even a lattice order).

Note that these algebras (unital implicative-magmas) are particular cases of *implicative-magmas*, where the existence of the unit 1 is not compulsory (i.e., an element 1 may exist, but it does not verify property (M)); consider, for example, *d-algebras*.

More generally, we have the 'world' of *non-commutative algebras of logic* (called here *unital pseudo-implicative-magmas*)

$$(A; \to, \rightsquigarrow, 1),$$

where there are (essentially) two implications, $\to$ and $\rightsquigarrow$, and an element 1 verifying property (pM):

$$1 \to x = x = 1 \rightsquigarrow x.$$

Here, there are two types of algebras: (i) algebras with a unique binary relation:

$$x \leq y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (x \to y = 1 \iff x \rightsquigarrow y = 1),$$

called *pseudo-algebras*—for example, *pseudo-BCI algebras* and *pseudo-BCK algebras*, recalled in Chapter 1—and (ii) algebras with two different binary relations:

$$x \ll y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad x \to y = 1$$

and

$$x \leq y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad x \rightsquigarrow y = 1$$

—for example, *semi-BCI algebras* and *semi-BCK algebras*, recalled in Chapter 1 as well. Note that these non-commutative algebras are particular cases of *pseudo-implicative-magmas*, where the existence of the unit 1 is not compulsory; consider, for example, *quantum-B algebras*, recalled in Chapter 1, where an element 1 does not exist in general, or *pseudo-d-algebras*, where an element 1 exists, but not verifying (pM).

(2) The 'world' of *unital commutative magmas*

$$(A; \odot, 1),$$

where there are (essentially) a product, $\odot$, and an element 1 verifying the properties (PU):

$$1 \odot x = x = x \odot 1$$

and (Pcomm):

$$x \odot y = y \odot x.$$

Here are some algebras belonging to this 'world': *groups*, *Boolean algebras*, *residuated lattices*, *MV algebras*, *divisible residuated lattices* (also called *divisible integral, residuated, commutative l-monoids*), *BL algebras*, *MTL algebras*, *WNM*, *IMTL* and *NM algebras*.

In 2020, the author introduced *m-MEL, m-BE, m-BCK algebras*, etc. ('m' coming from 'magma'), as analogues of MEL, BE, BCK algebras, etc. from the 'world' of commutative algebras of logic. They are unital commutative magmas with an additional operation, of the form

$$(A; \odot, {}^-, 1),$$

with $0 \stackrel{\text{def}}{=} 1^-$, where 1 is the *last element*, verifying (or not) (Pass) (associativity of product):

$$x \odot (y \odot z) = (x \odot y) \odot z;$$

and an internal binary relation can be defined by:

$$x \leq_m y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad x \odot y^- = 0$$

($\leq_m$ can be a pre-order, an order, or even a lattice order). Thus, the author obtains a new framework centered on m-BCK algebras. The class of m-BCK algebras (which are always involutive) contains those of MV, IMTL, NM, and Boolean algebras.

Then, the author 'incorporates' in the new framework all the quantum structures/algebras (i.e., putting them on the 'map'): *bounded involutive lattices*, *De Morgan algebras* and *ortholattices*, *quantum-MV algebras*, and *orthomodular lattices*. The connections between algebras of logic/algebras and quantum algebras, which were not very clear before, are thus clarified: it is proved that these quantum algebras belong, in fact, to the 'world' of *involutive unital commutative magmas*.

Note that these algebras (unital commutative magmas) are particular cases of *commutative magmas*, where the unit 1 is no longer compulsory (i.e., an element 1 may exist, but not verifying (PU)); consider, for example, *commutative semigroups*.

More generally, we have the ‘world’ of *unital magmas*

$$(A; \odot, 1),$$

where there are (essentially) a product, $\odot$, and an element 1 verifying the property (PU):

$$1 \odot x = x = x \odot 1;$$

consider, for example, *monoids*, which verify (PU) and (Pass). Note that unital magmas are particular cases of *magmas*, where the unit 1 is no longer compulsory; consider, for example, *semigroups*.

Between these two parallel ‘worlds’ there are some connections, ‘bridge theorems’, as for example: the equivalence between BCK(P’) algebras and pocrims, in the non-involutive case, and the definitional equivalence between Wajsberg algebras and MV algebras, in the involutive case ($(x^-)^- = x$). Two general ‘bridge theorems’ introduced by the author—recalled in Chapter 17—connect the two ‘worlds’ in the involutive commutative case, by the inverse maps Φ:

$$x \odot y \stackrel{\text{def}}{=} (x \to y^-)^-$$

and Ψ:

$$x \to y \stackrel{\text{def}}{=} (x \odot y^-)^-.$$

These theorems can be used to prove the *definitional equivalence* (d.e.) between the analogous involutive algebras from the two “worlds” simply by choosing appropriate equivalent definitions of these algebras.

This monograph of 690 pages has 17 chapters, divided into three parts: Part I (Pseudo-implicative-magmas. Implicative-magmas) (including the ‘world’ of BCK algebras) contains Chapters 1–4; Part II (Magmas. Commutative magmas) (including the ‘world’ of m-BCK algebras) contains Chapters 5–15; Part III contains ‘bridge theorems’ between the two ‘worlds’ (Chapters 16–17).

The book gathers mainly recent published results of the author. There are also incorporated in the book some new results, not published previously: some new results on L algebras, in Section 3.5, and very important final results on quantum structures, with many examples, in Chapter 15. The new results related to quantum-B algebras motivated the author to consider also the *non-commutative case*, in Chapters 1, 5 (Sections 5.1, 5.3–5.5) and 16.

The author says that this monograph would not have been written in so little time (2019–2022) and with so many important results and examples without the help of the computing program *Prover9/Mace4*, created by William W. McCune (1953–2011).

The monograph is written in a unifying way, which consists in fixing unique names for the defining properties, making lists of these properties and then using them for defining the different classes of algebras and for obtaining results. The author considers that, in a book of mathematics, a ‘map’, i.e., a figure showing the connections between the elements of an algebra or between different algebras, says more that thousand words; therefore, in this monograph, there are many figures. *Lavinia Corina Ciungu*