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MATHEMATICAL REVIEWS

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Bell, John L.

★**The axiom of choice.**

Studies in Logic (London), 22.
Mathematical Logic and Foundations.
College Publications, London, 2009. xii+248 pp.
ISBN 978-1-904987-54-3

John Bell's monograph is an extended appreciation of "probably the most interesting axiom of mathematics". In the first part of his book (Chapters I–IV), the author writes from the perspective of the working mathematician. In Chapter I, he tells of the origins of the Axiom of Choice (AC) and gives a brief chronology of its history. Maximal Principles are the subject of the second chapter where Zorn's Lemma (ZL) is the leading example. Again the origins and chronology of several such principles—the first one was due to Hausdorff in 1909—are the main concern. Of course, AC and ZL are logically equivalent, that is, in classical logic, as the author reminds us in a footnote (a hint of revelations to come).

Do you want an unmeasurable set? Use AC. Do you want a basis for your infinite-dimensional vector space? Use ZL. Chapter III is crammed full of useful consequences of AC and ZL and also some of the author's favorite equivalent forms of AC and ZL. But at the end of this chapter the author shifts us to a set theory that operates within Intuitionistic Logic (IL), a logic without the Law of Excluded Middle (LEM). In this "constructive" setting, we are told ZL is remarkably weak: it doesn't imply AC, and it doesn't even imply some consequences of AC that are known to be (classically) weaker than AC (the Boolean Prime Ideal Theorem, for example). This strange new world will be the subject matter of Chapter V, The Axiom of Choice and Intuitionistic Logic. (There is a brief appendix on IL.) But before that, Chapter IV contains sketches of proofs of the consistency of AC and $\neg$AC with Zermelo-Fraenkel Set Theory (ZF), that of ZF + AC by means of the inner model of hereditarily ordinal definable sets and that of ZF + $\neg$AC by means of a Boolean-valued model.

In the last three chapters the author changes his perspective. Chapters V and VII are written from the viewpoint of the constructive mathematician pondering the role of AC in foundations. As remarked above, Chapter V assumes IL as its background logic, and in the author's opinion it is in this realm that "the true depth of the connection between AC and logic emerges". For example, with the addition to IL of some rather weak assumptions, AC can be used to derive LEM. In Chapter VII, further constraints are imposed, and we are introduced to Constructive Type Theory, where again AC's role is examined. In Chapter VI , the in-between chapter, the author considers AC from the perspective of the topos theorist. (There is an appendix on the basic concepts of category theory.)

This book provides an interesting overview of AC from multiple perspectives; the pace is fast and in the latter chapters the reader meets some rather novel—to the traditional mathematician/logician—settings in which to study the import of AC. *J. M. Plotkin* (East Lansing, MI)