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★**Essays on set theory.**
Studies in Logic (London), 89.
Mathematical Logic and Foundations.
College Publications, [*London*], [2021], ©2021 . 593 *pp. ISBN* 978-1-84890-357-9

Beyond his own research work, the author of this book is well known because of his by now classic text *The higher infinite: large cardinals in set theory from their beginnings* [Perspect. Math. Logic, Springer, Berlin, 1994; MR1321144], and also as a co-editor, together with M. Foreman, of the three-volume *Handbook of set theory* [Springer, Dordrecht, 2010; MR2768678]. He has also written numerous essays on various set-theoretic topics.

This beautiful and inspiring book consists of nineteen of these essays, published between 1978 and 2016. In the introduction the author mentions seven more essays that were not included in the book.

The author has arranged the nineteen essays into four parts, called I. History, II. Philosophy, III. Mathematics, and IV. Lives in Set Theory.

The titles of the five essays in Part I, History, are:

1. "The emergence of descriptive set theory", reprinted from [in *From Dedekind to Gödel (Boston, MA, 1992)*, 241–262, Synthese Lib., 251, Kluwer Acad. Publ., Dordrecht, 1995; MR1746501],

2. (written together with Burton Dreben) "Hilbert and set theory", reprint of [Synthese **110** (1997), no. 1, 77–125; MR1451405],

3. "The mathematical import of Zermelo's well-ordering theorem", reprint of [Bull. Symbolic Logic **3** (1997), no. 3, 281–311; MR1476759],

4. "The empty set, the singleton, and the ordered pair", reprint of [Bull. Symbolic Logic **9** (2003), no. 3, 273–298; MR2005951], and

5. "In praise of replacement", reprint of [Bull. Symbolic Logic **18** (2012), no. 1, 46–90; MR2798268].

Part II, Philosophy, contains the following five essays:

6. "The mathematical infinite as a matter of method", reprint of [Ann. Japan Assoc. Philos. Sci. **20** (2012), 3–15; MR2957061],

7. "Mathematical knowledge: motley and complexity of proof", reprint of [Ann. Japan Assoc. Philos. Sci. **21** (2013), 21–35; MR3103444],

8. "Aspect-perception and the history of mathematics", reprint of [in *Aspect perception After Wittgenstein*, 109–130, Routledge, New York, 2017],

9. "Putnam's constructivization argument", reprint of [in *Hilary Putnam on logic and mathematics*, 235–247, Outst. Contrib. Log., 9, Springer, Cham, 2018; MR3890257], and

10. "Kreisel and Wittgenstein", reprint of [in *Kreisel's interests—on the foundations of logic and mathematics*, 1–32, Tributes, 41, Coll. Publ., [London], 2020; MR4241038].

Part III, Mathematics, contains three research papers, "the most expository and conceptually substantial so as to be plausibly termed 'essays'":

11. (written together with Robert M. Solovay and William N. Reinhardt) "Strong axioms of infinity and elementary embeddings", reprint of [Ann. Math. Logic **13** (1978), no. 1, 73–116; MR0482431],

12. (written together with Kenneth McAloon) "On Gödel incompleteness and finite combinatorics", reprint of [Ann. Pure Appl. Logic **33** (1987), no. 1, 23–41; MR0870685], and

13. "Regressive partition relations, n-subtle cardinals, and Borel diagonalization", reprint of [Ann. Pure Appl. Logic **52** (1991), no. 1-2, 65–77; MR1104054].

Finally, there are six essays in Part IV, Lives in Set Theory.

14. "Gödel and set theory", reprint of [Bull. Symbolic Logic **13** (2007), no. 2, 153–188; MR2322078],

15. "Levy and set theory", reprint of [Ann. Pure Appl. Logic **140** (2006), no. 1-3, 233–252; MR2224058],

16. "Cohen and set theory", reprint of [Bull. Symbolic Logic **14** (2008), no. 3, 351–378; MR2440597],

17. "Kunen and set theory", reprint of [Topology Appl. **158** (2011), no. 18, 2446–2459; MR2847314],

18. "Laver and set theory", reprint of [Arch. Math. Logic **55** (2016), no. 1-2, 133–164; MR3453582], and

19. "Mathias and set theory", reprint of [MLQ Math. Log. Q. **62** (2016), no. 3, 278–294; MR3509710].

Essays 1–5 should be read in connection with the author's important paper "The mathematical development of set theory from Cantor to Cohen" [Bull. Symbolic Logic **2** (1996), no. 1, 1–71; MR1380824].

The axiom schema of replacement distinguishes ZF from Z. The very informative essay 5 traces back the idea behind the axiom, called by the author a "bulwark of *indifference to identification*", to the work of Dedekind, who started to consider equivalence classes of a given equivalence relation as new mathematical objects and defining arithmetical functions by recursion. He sketches the history of the axiom and explains its origin in the work of Mirimanoff and Skolem and its crucial rôle in the work of von Neumann and Gödel. He also mentions opposition to the schema, by Fraenkel himself, for instance, and by Boolos, who objected to its consequence $\exists\kappa\,(\kappa = \aleph_\kappa)$. He of course reminds us of the use of the axiom schema in Martin's proof of Borel determinacy.

In essay 6, the author argues that, as mathematicians, we should consider how we handle the infinite in our arguments, constructions and proofs rather than get entangled in metaphysical discussions on the nature of the infinite. He advocates an *ecumenical* approach to constructive mathematics, as its reasonings are perfectly understandable and also part of the attempt to come to terms with the infinite.

In essay 7, the author discusses the role of proofs and formalization in mathematics, illustrating his argument by considering a number of famous more or less recent results.

In essay 8, the author defends *aspect-perception*, a notion due to Wittgenstein, as being useful for the study of mathematics and its history. He reconsiders the proof of the irrationality of $\sqrt{n}$ where n is not a square, an argument with both geometric and arithmetical aspects. One may consider examples, like $n = 2, n = 3, \ldots, n = 17$. The famous question, arising from Plato's dialogue, is how Theaetetus obtained the general result, as Socrates says, "at one stroke". The author claims to have found out how Theaetetus may have done it.

The author also considers the calculus textbook result that the cosine function is the derivative of the sine function. This fact has both geometric (Archimedes) and analytic (Newton) aspects, and the author observes that many textbook treatments contain a circularity that may and should be avoided.

In essay 9 the author considers Putnam's argument that Gödel's comment "$V = L$ is not *really* true" makes no sense. His main focus is on the mathematical results lying at the basis of Putnam's argument.

In essay 10, the author concludes that Kreisel's early conversations with Wittgenstein had a lasting influence on his interest in constructivity and proofs.

Essay 11 is the oldest and longest essay in the book. It is a seminal contribution to the theory of large cardinals.

Essay 12, inspired by the Paris-Harrington theorem, is an elegant and efficient presentation of a closely related combinatorial result independent from Peano arithmetic.

In essay 13 some of the ideas from the previous essay are generalized to provide a treatment of H. Friedman's Borel diagonalization propositions. The author characterizes their consistency strength in terms of n-subtle cardinals.

In the last six essays the author pays tribute to six of his brothers in arms. He tries to explain their specific contributions and how they changed set theory and our perspective of the subject. Of course it is a difficult exercise to delineate precisely the greatness of others, and one that only someone with the wisdom and experience of the author may bring to a successful conclusion.

The study of each of these essays is immensely rewarding. They are written with great care and make the reader profit from the deep and extensive knowledge of the author, who really writes *Ideengeschichte*, offering many new insights into the development of the great adventure of mathematics and set theory in particular.

We may be very glad they have been put together in this book. *Wim Veldman*