*Clippings from Issue 2009d*

# MATHEMATICAL REVIEWS

Established in 1940

**MR2404526 (2009d:03003)** 03-02 03-01 03F30 03F40

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## ★Incompleteness in the land of sets.

Studies in Logic (London), 5.
*College Publications, London*, 2007. xii+142 pp.
ISBN 978-1-904987-34-5

The hereditarily finite sets, $\mathbb{HF}$, are the medium through which the author presents an exposition of the classic meta-theorems that reveal the complexity of truth in $\mathbb{HF}$ and the limitations inherent in the axiomatic approximations to that truth. He provides a thorough introduction to the basic incompleteness and undecidability results of mathematical logic in the context of set theory. Aside from the requisite "mathematical ability", the reader should be familiar with mathematical logic through the Gödel completeness theorem. (Some acquaintance with set theory would also help.)

The first-order structure $\mathbb{HF} = \langle R_\omega, \in, A, \varnothing\rangle$, where $R_\omega$ comprises the well-founded sets of rank $< \omega$, $\in$ is the standard membership relation, $A(x, y)$ is the operation $x \cup \{y\}$, and $\varnothing$ is the empty set, can be viewed as the home of all the finitary mathematical objects needed to prove such results as Tarski's theorem on the undefinability of truth and Gödel's first incompleteness theorem. The first-order language relevant for $\mathbb{HF}$ is called $LS$. All the syntactical objects of $LS$, modulo an elementary coding, live in $\mathbb{HF}$. This allows the author to ask the crucial question: If $\varphi_S$ is the set defined by $\varphi$ in $\mathbb{HF}$, does $\varphi \in \varphi_S$? Calling a formula with one free variable "ordinary" if the answer to the crucial question is "No", a Russell's-Paradox-like argument yields that the set of ordinary formulas is undefinable (not representable) in $\mathbb{HF}$. In short order, this yields a version of Tarski's theorem: the set of sentences of $LS$ that are true in $\mathbb{HF}$ is not representable. This same idea leads to proofs of the well-known theorems of Gödel, Rosser, Church, and Post. The author approaches computability through definability. Sets defined by $\Delta_0$ formulas, those where all quantifications are bounded, are considered "constructive" in nature. Those defined by $\Sigma$ formulas are r.e. Arguments for these identifications are informal.

The hereditarily finite sets also have the following nice property: the standard model of arithmetic $\mathbb{N}$, with domain $\omega$ (the finite von Neumann ordinals), resides within $\mathbb{HF}$. Furthermore, there is a clever isomorphism from $\omega$ to $R_\omega$, which is due to Ackermann. The inverse of this mapping provides a comprehensive Gödel numbering of all the elements of $\mathbb{HF}$. This numbering allows the author to uniformly set his arguments in $\mathbb{HF}$ and then transfer them to $\mathbb{N}$.

The book ends with a proof of Gödel's second incompleteness theorem; in the proof, Löb's provability conditions are assumed and a suggestive modal notation is introduced. The latter leads to a very brief introduction to the provability logic GL.

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